

HIGHT

Inductance of Transformers

Electrical Engineering

M. S.

1911

UNIVERSITY OF ILLINOIS
LIBRARY

Class
1911

Book
H53

Volume



INDUCTANCE OF TRANSFORMERS

BY

938
52
u. o. / 3

EUGENE STUART HIGHT

B. S. UNIVERSITY OF ILLINOIS, 1910

THESIS

Submitted in Partial Fulfillment of the Requirements for the

Degree of

MASTER OF SCIENCE

IN ELECTRICAL ENGINEERING

IN

THE GRADUATE SCHOOL
OF THE
UNIVERSITY OF ILLINOIS

1911

UNIVERSITY OF ILLINOIS
THE GRADUATE SCHOOL

May 31

1961

I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

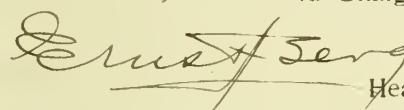
Eugene Stuart Hight

ENTITLED Inductance of Transformers

BE ACCEPTED AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE

DEGREE OF Master of Science in Electrical Engineering

 in Charge of Major Work

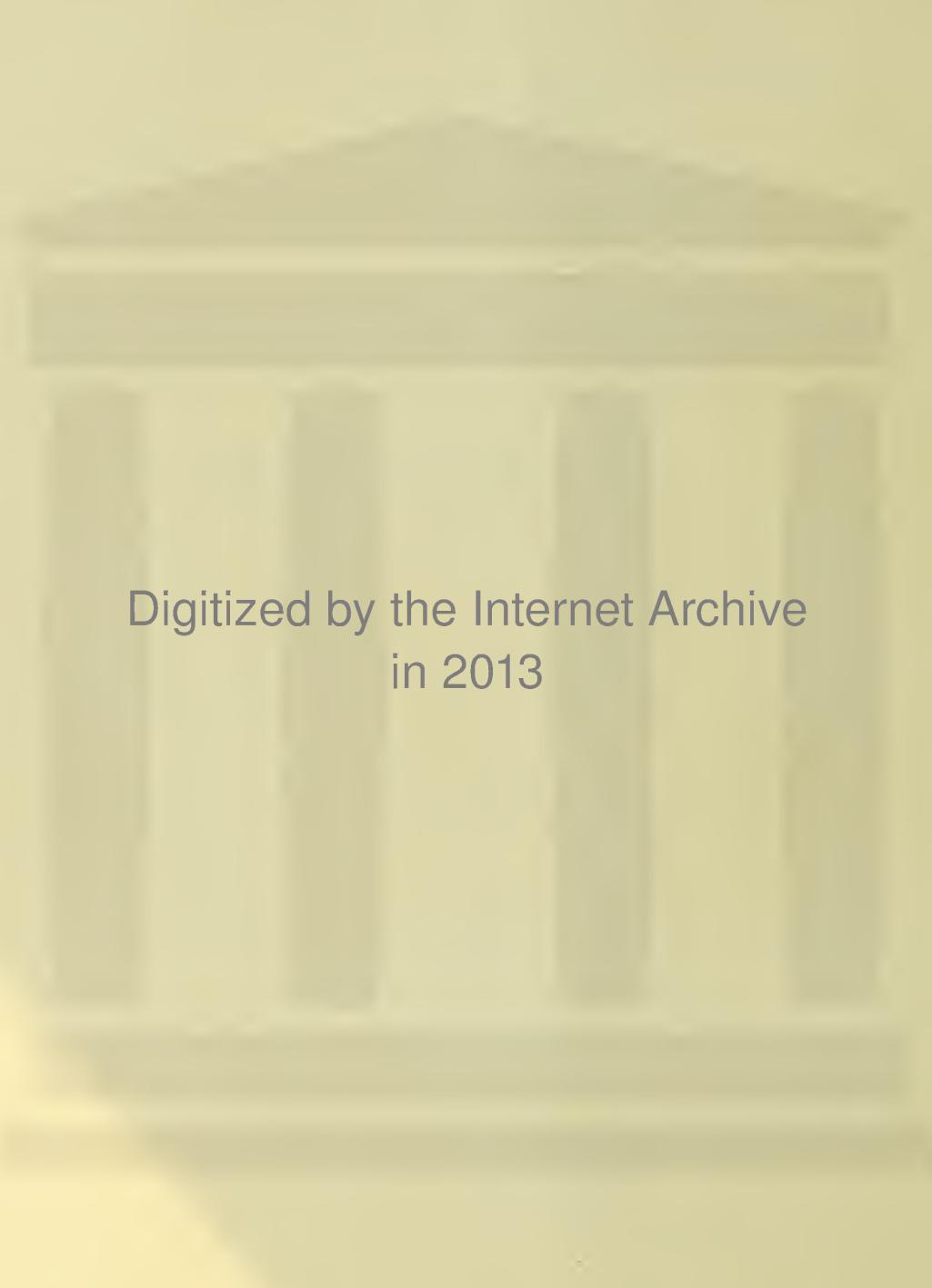
 Head of Department

Recommendation concurred in:

} Committee
on
Final Examination

197637

C O N T E N T S



Digitized by the Internet Archive
in 2013

<http://archive.org/details/inductanceoftran00high>

Table of Contents

-oo-

		<u>Page</u>
<u>Preface</u>	- - - - -	---
<u>Discussion</u>	- - - - -	---
Theoretical Discussion	- - - - -	---
Introduction	- - - - -	1
Theory	- - - - -	9
Tubes of Unit Stress	- - - - -	9
Stress Fields around Conductors	- - - - -	25
Distribution of Magnetic Stresses	- - - - -	28
Formula for Inductance of	- - - - -	
Air Core Coils	- - - - -	58
Formula for Leakage Inductance of		
Two Air Core Coils	- - - - -	62
The Core Type Transformer	- - - - -	65
Formula for Inductance	- - - - -	70
The Shell Type Transformer	- - - - -	70
Formula for Inductance	- - - - -	73
The Core Type Transformer with		
Flat Coils	- - - - -	73
Formula for Inductance	- - - - -	77
The Short Circuit Relations of		
Inductance	- - - - -	78

					<u>Page</u>
Theoretical Conclusions	-	-	-	-	82
Experimental Discussion	-	-	-	-	93
Description of Apparatus	-	-	-	-	94
Description of Tests	-	-	-	-	101
Results and Experimental					
Conclusions	-	-	-	-	107
<u>General Conclusions</u>	-	-	-	-	112
<u>Appendix</u>	-	-	-	-	116

* * * * *

P R E F A C E

P R E F A C E.

The general question of the Inductance of Transformers has been the basis of a very large number of investigations, some of which have resulted in valuable additions to the knowledge of the subject, already gained, while a larger number have become only stepping stones by means of which some explanation of an interesting, and seemingly inexplicable, phenomenon has been attained. It can not be claimed for the latter that they are peculiarly valuable, and yet, in a large number of cases, some really important discovery has depended upon the "mite" offered by one of the apparently valueless productions. Of these investigators, a large number have worked in Germany, some in England, and a smaller number in the United States; and, undoubtedly, all of them have in some way, given at least an increment of knowledge to the fund already at hand, concerning the subject.

To attempt, in a comparatively short space of time, to add very materially to the results already obtained, would be to fail almost before a start was made; but, undoubtedly, some small "increment of knowledge" concerning magnetic and the inductance of transformers, can be gained in even such a short time. The following paper offers the results of an attempt to gain such an increment.

Since, as was mentioned above, investigations have been carried on by a large number of men of science, there is available for general utility, a great amount of information concerning the

most fundamental characteristics of magnetic-electrical effects, the phenomena of iron circuits, and the application of these to the general methods of transformer construction. Consequently, it is assumed that the reader is familiar with a great many details such as the fundamental theory of the magnetic effects of electric currents, the construction of coils, the different types of the modern transformer, and like matters. In order to properly apply these details to the investigation of the magnetic leakage, and of the inductance of transformers as they actually exist, it has been thought necessary to include a rather detailed discussion of the theory of magnetic stresses in a nonconducting medium of uniform permeability, and of the application of this theory to coils having air cores, or cores of the same permeability as that of the surrounding medium. Only by assuming that the reader is equipped with a knowledge of the details mentioned above, is the development of this discussion in so small a space, made possible.

It is expected also, that this discussion will make logical, the somewhat involved assumptions which are utilized in the determination of the formulae for the different types of transformers. While great care must be used in making assumptions of this kind, it has been decided, after careful consideration, that for practical applications, they are justified. Their use brings up the question of the practical utility of involved theoretical formulae, the answer to which is evident. Still, the utilization of certain assumptions follows directly from the theory of air coils and the knowledge of the magnetic behavior of iron circuits; and for this reason also, in part, the discussion is given.

It will be noted that in this discussion, the part of the surrounding medium in the vicinity of a "live" conductor is

considered to be magnetically stressed, and considerably less emphasis is placed upon the theory of the "lines of magnetic force". It is believed that the conception of a field of stresses, simplifies the theory of magnetism and makes more directly applicable to coils, et cetera, the fundamental theory of the magnetic effect of an electric current.

In the preparation of the paper, a number of articles bearing upon the subject in general, were consulted. From three of these articles, ideas were obtained which are embodied in the development. Acknowledgement is consequently made to the following publications: "Elements of Electrical Engineering, Volume 2", Franklin and Esty; "Spannungsabfall und Streuung der Transformatoren", Dr. Gustav Bernischke, Elektrotechnische Zeitschrift, 1908; "Die Streuung bei Wechsel-Transformatoren und Kommutator-Motoren", Dipl. Ing. W. Rogowski und Dr. K. Simons, Elektrotechnische Zeitschrift, 1908.

May, 1911.

Eugene Higkt.

D I S C U S S I O N

T H E O R E T I C A L

I N D U C T A N C E O F T R A N S F O R M E R S

At the present time, the progress of the practical side of the Science of Electrical Engineering is toward the concentration of generating equipment. The engineers who have electrical development in charge, are considering seriously, the problem of the isolated central station and its small area of power consumption, as compared with the large generating station, located in such a place that energy distribution over a large area is feasible, while at the same time, the operating economies may be increased. The use of coal and water makes necessary at times, very careful consideration of such location, and often, as a result, the chosen spot at first appears to be so very far from any logical point of power consumption that the attempt to prepare the plant for operation in such a place, would be the extreme of folly. And yet, the engineers are not so far wrong. The rapid development of the generation of alternating current, of the transmission, by means of it, of enormous quantities of energy at very high voltages, and of the distribution of this energy at voltages suitable for general use, has made the large centralized generating station an important factor in the progress of the utilization of electrical energy.

In the operation of such a centralized station, the transmission and distribution of energy are factors of importance

equal to that of the generation of the energy, itself. In fact, the distribution equipment alone, neglecting the transmission line, is of utmost importance; since it is due to the constantly efficient and economical operation of this equipment, that the project as a whole, can be successfully operated. The generating station, carrying as it does, the total load of the distribution systems which are connected to it, operates under approximately constant conditions. The load, the voltage, and the power factor, are practically uniform thruout any period of time. In the case of the transmission line, also, such conditions obtain. However, at any one of the comparatively small distributing stations, the load is almost continually changing, and with it, unless some method of regulation is present, change the voltage and the power factor. Thus, while any one unit of the distributing equipment is not in operation as continually as is a generating unit, it is subjected to the rigorous task of handling a varying load, which, while it does not, perhaps, injure the equipment, may, by its effect upon the uniformity of operation, cause such poor service to be rendered that the business department of the project will be seriously affected. It is due, to a certain extent, to influences of this nature, that the distribution equipment must be considered to be of such importance and must be given so much attention.

There is however, another characteristic of the operation of the distributing equipment which has an influence upon the economies of the whole installation, even greater than that of the variations of voltage and efficiency. For, while these latter are of utmost importance from the point of view of the customer, or of the user of the distributed energy, and as a consequence, must occupy some of the attention of the operating engineer, the actual process

of operating the distributing equipment, considered from the point of view of the operating company, becomes a powerful factor, if serious injury to this equipment, because of some inherent and peculiar characteristic can result under ordinary operating conditions. Such a characteristic as this, is embodied in all alternating current generating and transforming machinery and is called "reactance". The percent reactance of distributing equipment becomes of particular importance when transforming machines, such as transformers of the ordinary type, are operated in parallel; and since such operation forms a large proportion of the general operating schemes of very large, and also, of those of even moderately large, systems, it is evident that the consideration of this characteristic is of importance to the operating engineers in charge. The particular effect which makes this reactance of such importance, is that upon the division of the load among transformers operated in parallel. If, for example, two such machines operating in this way, are each to carry half of the connected load, and thru mistake of the manufacturer or of the installing engineer, the percent reactance is different in the two machines, the load would not be equally divided and the division would be the more unequal, the greater the difference in the percents reactance. Under actual conditions, the mistake of even a fraction of a percent in the determination of the reactance of a transformer might mean the impressing upon this machine of perhaps one hundred percent overload, which in a very short time would mean the destruction of the apparatus. For this reason, it is of the utmost importance to accurately determine the value of this characteristic for any machine to be used under such conditions; and such a determination becomes a greater factor than efficiency and regulation combined, in its effect upon the satisfactory operation

of a large system.

Because of such conditions, the operating engineers of these systems of distribution are demanding from the manufacturers, equipment which is very highly guaranteed in regard to satisfactory inherent operation as well as to voltage regulation and efficiency. In response to this demand, the manufacturers are endeavoring continually to better their product, and as a consequence, are discovering new problems of design and construction at every point of their progress. It is to the solution of a large number of these problems that the present highly developed equipment is due.

The distribution equipment which is the product of these manufacturing companies, consists chiefly of constant potential transformers. These machines are designed to receive energy at an approximately constant potential, and to deliver it at some approximately constant potential of different value. In the distributing circuits, the impressed voltage is high and of a value depending upon the distance of transmission; while the voltage at which the energy is delivered is of a value corresponding to the method of utilization of the energy. At the generating station however, the transformers receive energy at the generated potential, and deliver it to the feeders and to the transmission lines at the voltage desired for transmission. In the distributing system, there are transforming units for practically every consumer, no matter how isolated or how small is the energy consumption; while in the generating station, there are usually, only the transformers having large capacity and handling the large divisions of the output. Consequently, for each "step-up" transformer, there may be a hundred or many times a hundred "step-down" transformers of small capacity and of comparatively low ranges of potential. As a result of such conditions,

the manufacturers are confronted with the problem of producing an enormous quantity of distributing transformers, having an accurately predetermined value of reactance as well as low voltage regulation and high efficiency.

Into such predeterminations of reactance, regulation, and efficiency, enters a very important factor, the effect of the magnetic leakage, or of the stray lines of magnetic stress within the windings and the iron circuit of the transformer. Exactly how important this factor is, cannot be stated. It is sufficient, however, that to this phenomenon of stray magnetic fields, is due the percent reactance of the machine, as well as a large percent of the voltage regulation and a smaller percent of the losses.

An investigation of the influence of these leakage lines is opposed by many difficulties, which, perhaps, together with the failure to recognise the importance of the phenomenon, are accountable for the fact that a very small amount of definite knowledge is extant, concerning these unhappily strayed fields and their effects upon the operation of the transformer. It is the purpose of this paper to present a theoretical investigation of the behavior and effect of magnetic leakage, and to offer the results of a series of tests, in the endeavor to increase the knowledge of, and the respect for, this phenomenon and its effects.

In the development of the investigation, the prevailing idea will be, to present in as clear form as is possible and advisable, the theory of the iron circuit in its application to transforming equipment. In order to do this, it will be necessary to undertake first, a general discussion of the phenomena of flux or of the flow of lines of magnetic stress, and their effects. From this discussion, the application of the theory of flux to transformers will

be derived and particular emphasis will be placed upon the core and shell types of single phase transformers, since these types prevail thruout the electrical world. The results from the theory of the leakage flux will then be theoretically considered. Following this, will be given a discussion of the tests made and of the conclusions drawn from them. Finally, the general results of the investigation will be given. It is of course, expected that the development of the investigation along these lines will not result in an entirely complete and comprehensive discussion of magnetic leakage and of the inductance of transformers; but, that it will result in a furtherance of the general knowledge of the phenomenon and of the effects which are due to its presence.

Magnetic leakage is, primarily, a leakage or straying of lines of magnetic force from the paths which it is desired, they follow. A line of magnetic force is a line of magnetic stress thru the medium in which the line lies; or, it may also be considered as representing the axis of a tube of unit magnetic stress. This conception can well be illustrated in the following manner: Suppose a net, having meshes of some definite and uniform size, be stretched in a medium such as ether, or any other substance having uniform permeability. Also suppose another net, having meshes uniform and of the same size as, or of different size from, those in the first net, to be stretched at some distance from the first in the same medium. Now consider any one mesh in each net. Suppose that lines of a uniform field, each of which represents an increment of magnetic stress in the medium, originate, one from every point in the periphery of the chosen mesh in the first net; also, that these lines end, so far as this part of the medium is concerned, around the periphery of the chosen mesh in the second net. It is evident, that in

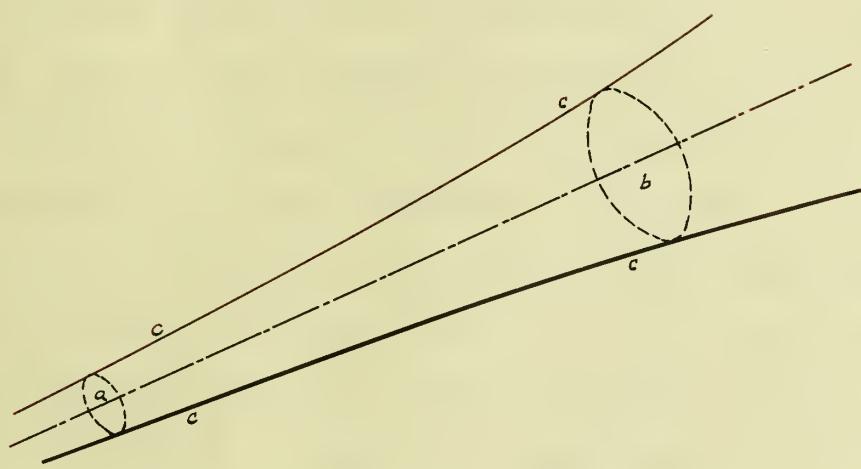


FIGURE 1.

their passage thru the medium lying between the nets, the lines unite to form the closed outer surface of a tube, which extends from one net to the other. Suppose that a, Figure 1, is the outline of the first chosen mesh, and b, that of the second chosen mesh; also that c,c,c, are single lines of increment stress, two of the infinite number of lines extending from the periphery of a to that of b. Now, since it is assumed that these are lines of force, they will lie in the direction of the general magnetic field of the medium. Then no lines of force will pass thru the walls of the tube c c, since all the other lines in the field are parallel to these walls. Consequently, all the lines passing in thru a will pass out thru b, and the flux or the flow of lines thru the interior of the tube, will be constant at all right sections of the tube. If now, the amount of flux or the number of increment lines, passing thru the area a, be considered as unity, that is, if the density in the area a, be unit density, and if the size of the mesh varies in such a way that where the field is weak, the area of b is large, so that the number of lines per unit of area will correspond to some other density which is a measure of the weakness of the field, then the tube may be considered to be a tube of unit magnetic stress, the effect of which is the summation of the effects of the infinite number of lines of increment stress which form it. Now if there is a corresponding tube for each mesh in the first net and in the second net, the resultant bundle of tubes across the medium will represent the distribution of magnetic stress in the medium; and if each one of these tubes is represented by a line passing thru the centers of its right sections, the resulting bundle of lines will represent the distribution of magnetic flow or flux thru the medium between the nets. It is convenient to speak of a unit tube as a line of magnetic

flux. Now suppose that, starting from a, the unit tube passes the second net at b, and after traversing a path of greater or less length, and suffering more or less change in cross sectional area to correspond to the change in permeability in the various media thru which it may pass, returns to a, arriving on the opposite surface of the net, thus forming a continuus circuit. Then suppose that all the tubes originating at the first net act in the same manner, passing thru various other media, having various characteristics under different circumstances, and finally, returning intact to the first net from the opposite side. These tubes will represent the distribution of magnetic stress thruout the medium in which they exist, and center lines drawn for all of them, will represent the distribution of the magnetic flux or flow thruout the same region. Such a distribution of unit tubes and magnetic lines is called a magnetic field.

There are then, certain definite characteristics of the lines of magnetic stress which must be briefly considered. They are as follows:

(a) Due to the necessity of preserving an equilibrium thruout the region occupied by a stationary field of magnetic stress, the lines of force are reentrant, or form a closed magnetic circuit, and at no instant is a line broken.

(b) Due to the conception of a tube of unit magnetic stress, the lines of force cannot cross each other; and the lines of any two fields cannot cross each other in the same region.

Such a field of stress, or for more general consideration, such a line of magnetic stress, may be produced by the action of an electric circuit. If for example, a straight wire carries a current of some value I, due to the magnetic effect of this cur-

rent, the medium surrounding the wire will be magnetically stressed and there will be set up about the wire and in planes parallel to its right section, tubes of unit stress, which may be represented by lines of magnetic force. Figure 2 represents one of these planes. The wire \textcircled{o} , carries a current of a constant value I , in a direction perpendicular to the plane of the paper and toward the latter from the reader. There are set up, about the wire and concentric with it, lines of magnetic force in the direction shown by the arrows. These lines are close together near the wire, where the density is higher due to the shorter path, and due also to a larger number of lines of increment stress per unit area, and hence smaller unit tubes to contain them. Farther from the wire, at \textcircled{a}' , the density is lower, due to the longer path, and due to less force available for producing the increment lines, and consequently, the lines of magnetic stress are farther apart, since the unit tubes are larger. The stressed condition of a volume increment of the medium surrounding the wire becomes less as the distance of the increment from the wire, is increased. Then at any such distance between zero and infinity, a given increment of volume has a definite condition of stress. As the distance from the wire increases, however, the stress becomes very, very small, and after certain distances are reached, depending upon the current flowing in the wire, the stressed condition becomes inappreciable and points so reached are said to be out of the magnetic field. Such in general, is the behavior of the magnetic field set up by a constant current flowing in the wire.

Now suppose the constant current to be destroyed suddenly, that is, the current value reduced to zero. Practically at once, the magnetic stresses set up in the surrounding medium, also fall to zero. The destruction of these stresses or of the magnetic field

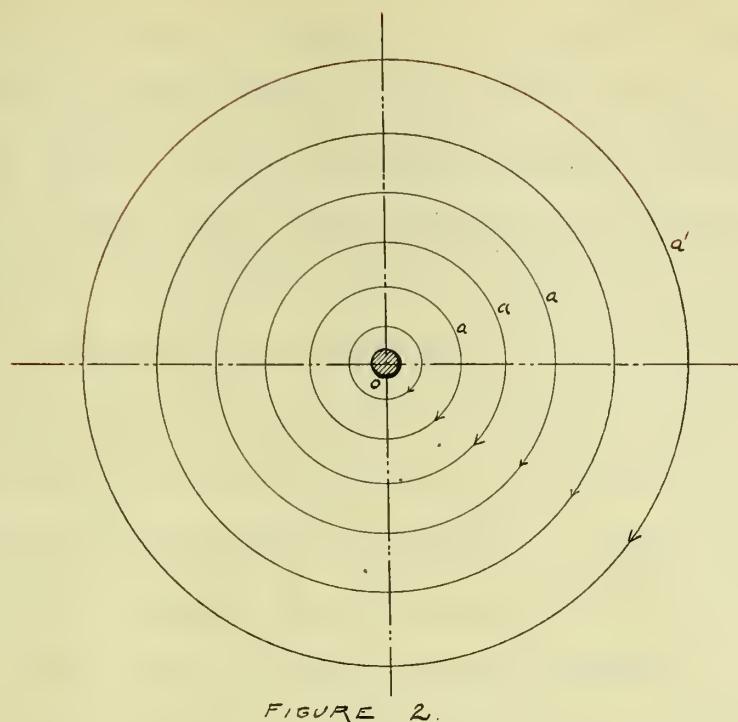


FIGURE 2.

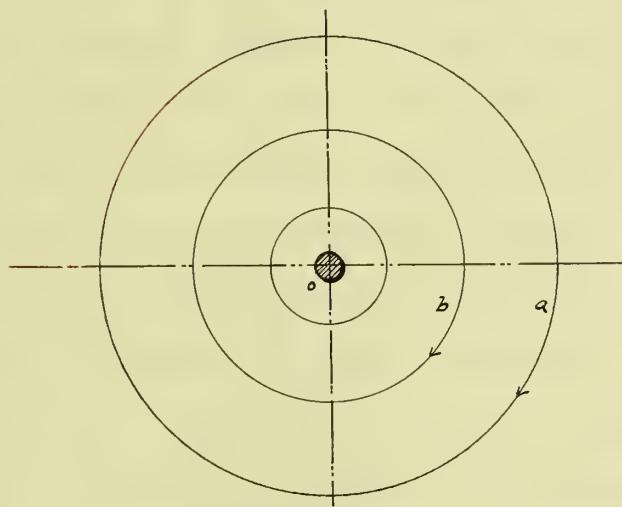


FIGURE 3.

is accomplished in this manner: Consider any line of stress a, Figure 3, representing a closed tube of magnetic stress set up around the wire o, by reason of the effect of the constant current flowing in the wire. Let this current suddenly be reduced to zero. Immediately, or very nearly so, the tube a, begins to shrink, or to become smaller in diameter both as a tube and as a closed circuit. Eventually, it reaches the positions b, and c, respectively, and finally, continually moving inward, resolves itself into a point circle in the wire at o, maintaining to the end, its characteristic of a closed tube of unit magnetic stress. After the tube has reached this condition, suppose the original current to be again established in the wire but this time, in the opposite direction. At its first appearance, the point circle, which is a remnant of a, begins to expand and reaches in succession, positions c, and b; and finally, when the current reaches the original value, assumes the original position at a, moving outward to this position in a plane perpendicular to the axis of the wire. When this condition becomes stable, suppose the current circuit again broken and again established in the direction which it possessed at first. Again the magnetic field disappears and is set up in the same condition as that in which it originally existed. If such a complete cyclic change takes place a large number of times in a second, the resultant changes in the magnetic field consist in a series of alternations of the magnetic stresses in the medium surrounding the wire. And if the current is gradually set up to, and withdrawn from, each maximum value, the magnetic stresses will be set up gradually and will gradually return to zero. This is the condition existing when an alternating current flows in the wire.

Hitherto, the wire under discussion has been considered

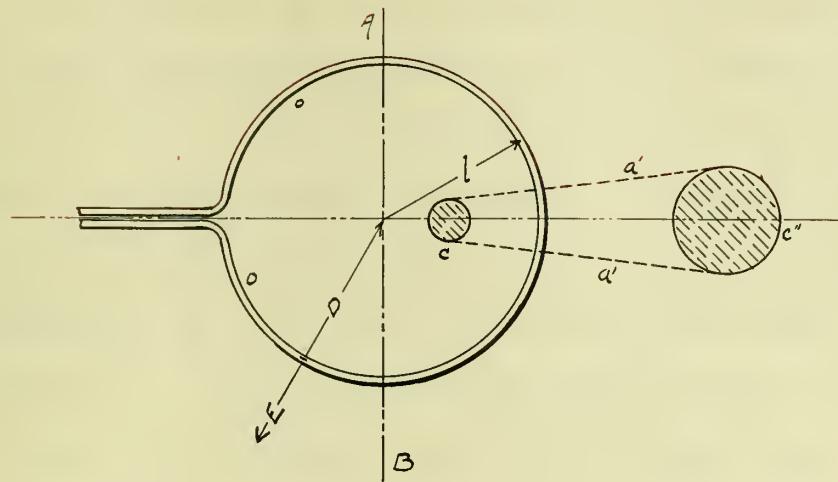


FIGURE 4.

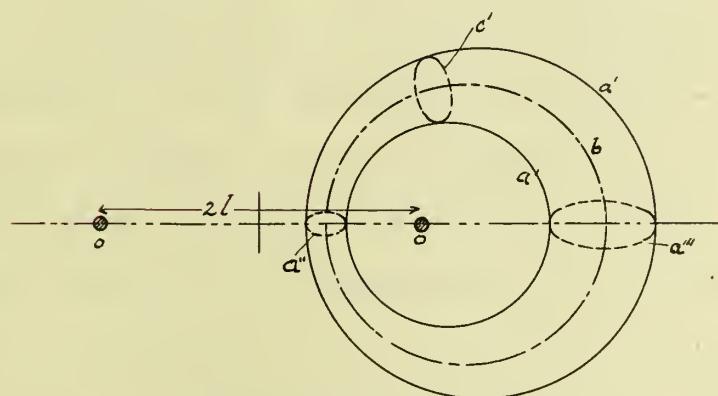


FIGURE 5.

as straight and the magnetic lines set up were those in any plane perpendicular to the axis of the wire. From the general discussion of the nature of a tube of stress, it is evident that any tube represented by a line a, or b, is of constant cross sectional area, thruout its length, provided of course, that the permeability or facility with which the unit magnetic stresses are set up, is the same in all parts of the medium. Consider now that the wire is bent so that it forms a plane circle o.o.o, as is shown in Figure 4. This circle has the radius l, and hence a circumference $2\pi l$. Consider also, that current of a definite constant value flows in the wire. It is evident from the discussion given above, that when the current starts to flow thru the wire, the tubes of unit stress tend to place themselves concentrically about the axis of the wire in a plane perpendicular to this axis. Now, if some one tube as a in Figure 3, placed itself, under the first condition, in a position at a distance from the wire o, greater than l, the same tube under the new conditions, must be distorted, and while the part outside the wire may extend to any position in the direction E, Figure 4, its position within the circle, that is, in the direction O, is limited to a distance from the wire of l, the radius of the circle, since the tubes of unit stress, set up in the same plane, but by the part of the wire diametrically opposite, divide, with the tube a, and the others originating with a, the part of the plane zl, lying between the right sections of the wire. Also, since there may be many increments of length around the circumference of the circle, each one of which sets up a series of tubes of unit stress in a plane parallel with its right section, it is evident that the right sections of the unit tubes in the plane of the circle, and within its circumference, are much smaller than those outside the circumference of the circle.

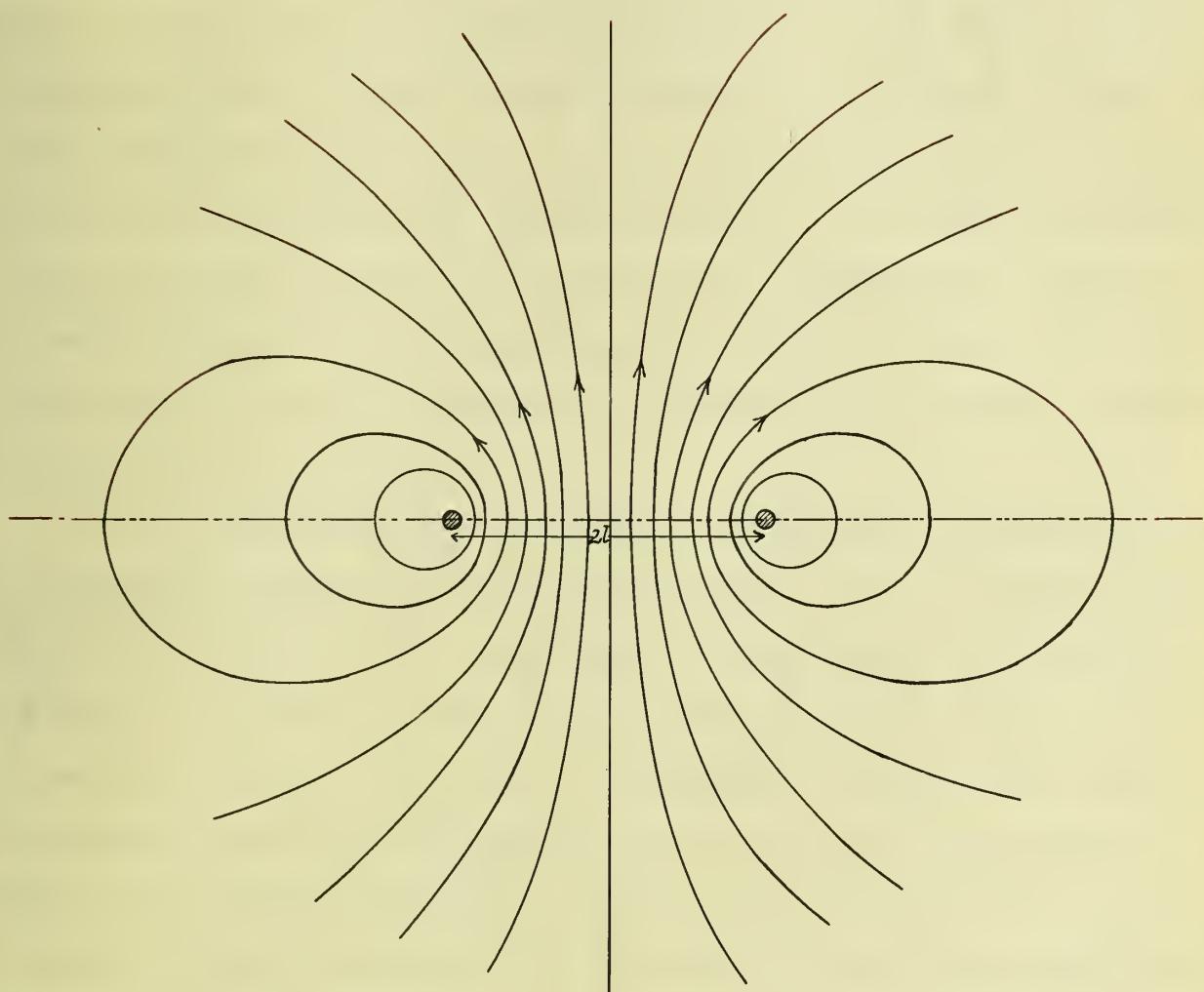


FIGURE 6.

Thus in Figure 5, a", a "", are the right sections of this tube. a" is the smaller, relatively, the greater the number of unit tubes in the field set up by the whole length of wire. If each tube in the plane of Figure 5, (which is a section along the line AB, Figure 4) is represented by its line, as b, the section of the field by the plane of Figure 5 is represented in Figure 6. Each line, e, e, e, represents a tube of unit magnetic stress. Outside the circle, the lines are farther apart due to the increased areas of the right sections of the unit tubes as explained above. If in this arrangement also, the current in the wire alternates, the magnetic stresses in the surrounding medium, alternate, from zero to a maximum and back to zero; while the unit tubes shrink to point circles, expand to normal size and return to zero.

When however, the current in the wire changes direction, the magnetic stresses also change direction. If, in Figures 7 and 8, the current flows as the conventional symbol shows, the direction of the magnetic stress is that of the curved arrows. So that, when the current alternates in the wire, the magnetic stress in the surrounding medium passes from a maximum in one direction, thru zero to a maximum in the opposite direction, and the unit tubes are set up, reduced to point circles, and again set up in the original position; having however, in the second case, an effect opposite to the one just preceding.

Such is the behavior of the fields of magnetic stress set up around a wire carrying an alternating current. It has been shown that the current, by its effect, sets up such a field. The field then, if due to the magnetic effect of the current, should have an electrical effect upon any electrical conductor which might come into its vicinity; since, due to the continual attempt of Nature to

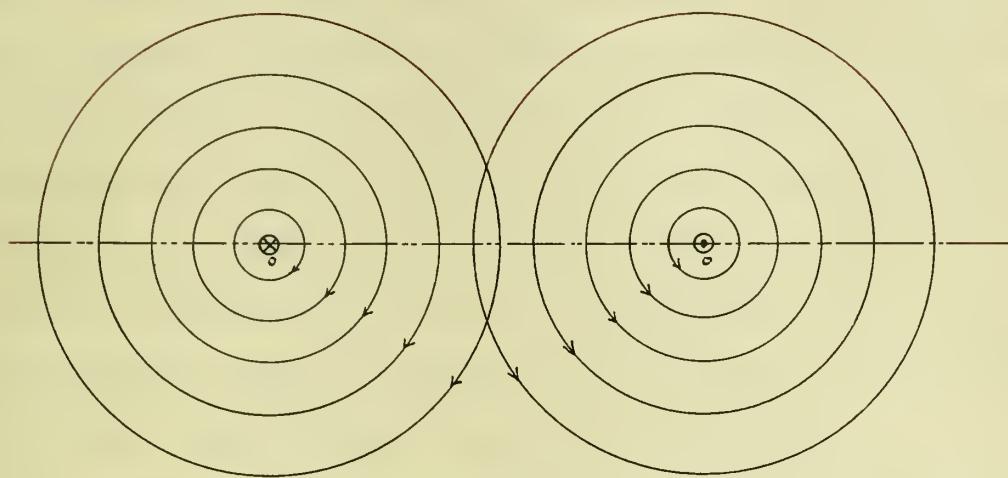


FIGURE 7.

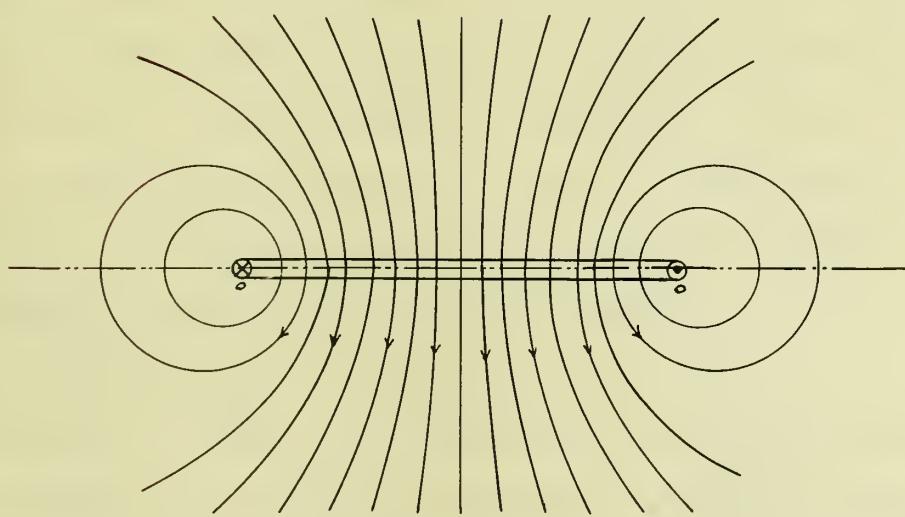


FIGURE 8.

preserve an equilibrium, the energy utilized in setting up the magnetic field should be transferred if, in any way, the equilibrium established after this field is set up should be destroyed, wholly or in part. In actual fact, such is the case. If an attempt is made to set up a magnetic stress in a medium, resistance is offered by the medium in proportion to the magnitude of the stress. And since the force producing the stress is due, in the resolution, to the electromotive force which causes the current to flow, the resistance of the medium to the setting up of a field of stress must be of the nature of an electromotive force in order to act against the original electromotive force. This effect takes place as follows:

Consider a wire lying in a medium of equal permeability thruout, and having no current flowing in it. The conditions of the wire and of the medium are those of equilibrium. Now if an electromotive force is impressed upon the wire, a current will attempt to flow thru the wire. At the first appearance of the current, the tubes of unit stress will appear and, as the current increases, the magnetic stresses in the medium will increase. As the original equilibrium is destroyed and while a new one is being produced under the new conditions, the medium resists the change by inducing in the wire, an electromotive force in a direction opposite at every instant, to that of the one existing. This induced electromotive force tends to cause a current to flow in the wire, and this current tends to set up a field of stress in a direction opposite to that of the field which the original current is producing. As soon as the original current has reached its full value and the field due to this current has reached its normal position in the medium, there is again a state of equilibrium and the induced electromotive force in the wire is zero, and consequently there is no opposing field of stress. If now, the

impressed electromotive force is reduced to zero or increased or changed quantitatively in any way, the equilibrium is again destroyed and an electromotive force is induced in the wire, which tends by its effect, to prevent the departure of the system from equilibrium.

Now, it is evident, that if an alternating electromotive force is impressed upon the wire, and an alternating current flows, the system will not be, at any time, in a state of equilibrium. The stress in the medium, passing from zero to a maximum in one direction, and again to zero, is continually opposed by a fictitious field, produced by the continual attempt to preserve an equilibrium. With alternating current, then, there is an electromotive force induced continually in the wire in opposition to the electromotive force impressed upon the wire. (Resistance assumed to be negligible). It is evident that the magnitude of this induced electromotive force is dependent upon the magnitude of the change in the stresses in the medium.

Since this electromotive force is induced in the wire carrying the current, it is known as the electromotive force of self-induction. Consider now a wire lying parallel to the original wire in the medium. Suppose an alternating current existent in the original wire, and that for consideration, the instant is chosen at which the magnetic stresses in the medium are increasing to a maximum. Consider some one unit tube represented by the line a, in Figure 9. As this tube moves to its position beyond the second wire o', it comes into contact with this wire. The electrical character of the obstruction is different from that of the original medium, (assuming the medium to be nonconductive, electrically), and as the tube crosses the wire, or rather, passes thru it, the equilibrium of the system, consisting of the tube a, the second wire o', and the original

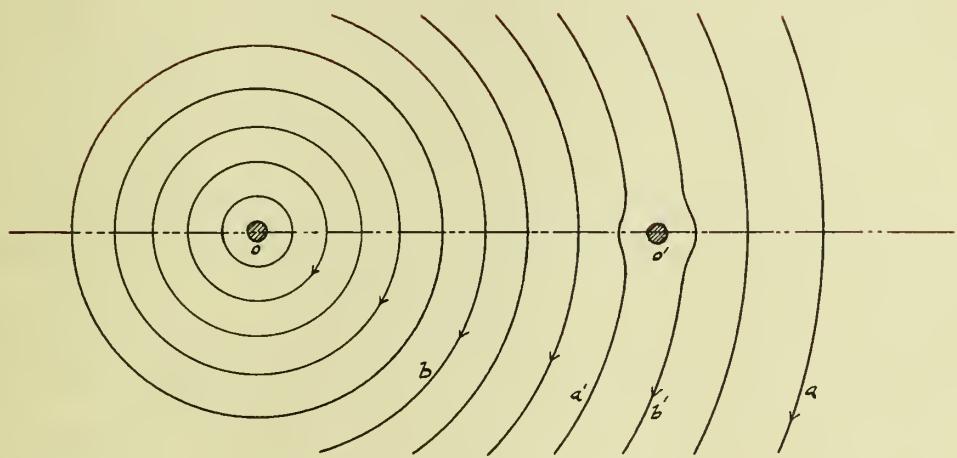


FIGURE 9.

medium, undergoes a certain change, although it is again established the instant after the tube has passed thru the wire σ' . An attempt is made to prevent this change by the induction of an electromotive force in the wire σ' . If this wire forms a closed circuit, having its return conductor outside the field of magnetic stresses due to the first wire, a current will flow, and this current will tend to prevent the instantaneous change of equilibrium due to the crossing of the line σ . If any other lines cross the wire, σ' , they will also each induce an increment of electromotive force in the wire. As soon as the equilibrium is established by the final setting up of the stresses due to the original current in σ , there is no more electromotive force induced in σ' , and consequently there is no field of stress due to a current in σ' . Due to the alternating current flowing in σ , however, the stresses due to this current are no more than set up in one direction before they are reduced to zero and established in the opposite direction. As the unit tubes pass to zero, they again induce an electromotive force in the wire, and this same phenomenon occurs each time they are set up or destroyed. Since however, when the unit tubes are set up during the second half of the cycle, the stresses are in a direction opposite to that in the first half of the cycle, the induced electromotive force in the wire σ' , is a single, and not a double frequency quantity. It is evident that this induced electromotive force, called the electromotive force of mutual induction, is of a value dependent upon the magnitude of the stresses set up in the medium due to the tubes of unit stress which pass across the second wire. That is, any unit tubes which may normally lie between the two wires, do not induce any electromotive force in σ' , since they do not aid in destroying any equilibrium by passing across σ' . It is evident also, that if the two wires were formed as plane

circles, the action and reaction of the magnetic stresses would correspond to this explanation, if the planes were parallel and the circles were concentric. The forms alone, of the unit tubes, would be different from those in the case of the parallel, straight conductors.

Consider now, that $\underline{o'}$ is part of a closed circuit. When the current, due to the induced electromotive force, appears, there will be a tendency for secondary lines of stress to be set up, in order to oppose the inducing stresses. Some of these secondary stresses will be set up. Then, in the same manner as was explained before, as the secondary current increases, the secondary stresses increasing, destroy a state of equilibrium in the secondary system, and for this reason cause an electromotive force of self induction to appear in the secondary wire $\underline{o'}$, which by its effect, tends to set up a series of magnetic stresses opposing those which appear and which are due to the true secondary current. Also, due to this true secondary current, there is a tendency for certain tubes of unit stress, as they pass out from the wire $\underline{o'}$, to pass across the wire \underline{o} , and by their effect to induce in \underline{o} an electromotive force of secondary mutual induction, which in turn, by its effect, tends to oppose these stresses due to the true secondary current. These characteristics of the system are evident from the discussion of the changes of equilibrium, due to the variation of the magnetic stresses in the medium.

From the foregoing, it may be deduced, that if two wires, each forming part of a closed circuit, are placed parallel to each other in a given medium, and if an alternating current flows in one circuit, there are four divisions of the fields of the magnetic stresses which surround the wires. These divisions are as follows:

- (1) The field of stress due to the primary or original in-

ducing circuit, lying between the wires, or interlinking with only the primary circuit.

(2) The field of stress due to the primary circuit, lying outside the second wire, or interlinking with both circuits.

(3) The field of stress due to the induced current in the second circuit and lying between the wires, or interlinking with only the secondary circuit.

(4) The field of stress due to the induced current in the secondary circuit and lying outside the primary wire, or interlinking with both circuits.

Of these four divisions of the field of stresses, (1) and (3) are the "singly interlinked fields"; while (2) and (4) are the "doubly interlinked fields". The singly interlinked fields are the leakage fields; the flux in these fields is leakage flux; and the magnetic stresses due to these fields are leakage stresses. It is evident that (1) has no effect upon the secondary circuit; and that (3) has no effect upon the primary circuit; and, due to these characteristics, the name "stray" or "leakage" fields, is given them. It will be noted also, that fields (1) and (2) are effective in producing the electromotive force of self induction in the primary; that (2) is effective in producing the electromotive force of primary mutual induction; that (3) and (4) are effective in producing the electromotive force of secondary self induction; and, that (4) is effective in producing the electromotive force of secondary mutual induction.

Such is the general qualitative behavior of the magnetic stresses set up in a medium due to the alternating current which flows in a wire in the medium. It is seen that each phenomenon is dependent upon surrounding conditions for its development. Since

this is the case, a law for each development may be stated, based upon the limiting conditions. Such laws are most easily stated mathematically, and since they are dependent, usually, upon conditions which may be mathematically expressed, they offer a means for the quantitative determination of the behavior of the magnetic stresses. Such determination will be next attempted.

Consider first, the condition of Figure 2; a straight wire lying in a medium of uniform permeability thruout. Let the maximum value of current be existent in the wire and suppose the current constant at this value. As was shown before, a field of magnetic stress is set up in the medium surrounding the wire. This field has the general form shown in Figure 10, in which the lines $\underline{\alpha}$, $\underline{\alpha}$, $\underline{\alpha}$, represent tubes of unit stress, existing in the medium. Consider two parallel plane surfaces passed thru the medium perpendicularly to the wire, and separated by a length $\Delta\beta$. Consider any circular cylinder concentric with the wire and having the radius $\underline{\alpha}$. A determination will be made showing the amount of stress set up in that part of the medium bounded by the cylindrical surface and the two plane surfaces. Suppose an annular ring of width Δx , and thickness $\Delta\beta$, lying between the two plane surfaces and within the circular cylinder, at a distance \underline{x} from the wire \underline{o} . Now the magnetic effect of the current in the wire, or the "magneto-motive force", is the product of the current flowing in the wire and the number of times this current acts or the number of conductors. Then, if F denotes this magneto-motive force, the ampere turns due to the current of \underline{I} amperes flowing in the wire \underline{o} , is

$$F = I$$

since \underline{N} , the number of turns,

is unity. Now, a C. G. S. unit of current is that value of current which, flowing in an electric circuit of the nature of the one under

consideration, produces, in a path around the conductor and concentric with, and at a unit distance from, the center of the conductor, two units of magnetic stress per unit of cross sectional area of the magnetic path, or a magnetic stress intensity of $\underline{2}$. Since the length of the path is $\underline{2\pi}$, the stress intensity in a path of unit length will be $\underline{2 \times 2\pi}$ units or $\underline{4\pi}$ units. Since an ampere is one tenth of a C.G.S. unit of current, it follows that an ampere produces in such a path of unit length, a stress intensity of $\underline{0.4\pi}$ units, per unit area of cross section of the path. Since the unit of area is a square centimeter, and the unit of length is a centimeter, an ampere of current flowing in a wire as shown in Figure 11, produces in a path of one centimeter length, a stress intensity of $\underline{0.4\pi}$ units, or unit tubes, or magnetic lines, or simply lines, per square centimeter. Then in a path of length \underline{l} centimeters, the stress intensity is $\underline{\frac{0.4\pi}{l}}$ and if a current of \underline{I} amperes flows in the wire, the stress intensity in this path is

$$\mathcal{H} = \frac{0.4\pi I}{l} \quad (2)$$

Now consider a path at the distance \underline{x} from the center of the wire \underline{o} . The length of this path is $\underline{2\pi x}$ centimeters; then the stress intensity in the path is

$$\mathcal{H} = \frac{0.4\pi I}{2\pi x} = \frac{0.2I}{x} \quad (3)$$

$(x \neq 0)$

Suppose the path to have the cross sectional area $\underline{\Delta p \Delta x}$; then the total stresses set up in the path, being an increment of the total stresses set up in that part of the medium lying, as in Figure 11, enclosed by the plane surfaces and the cylindrical surface, will be

$$\Delta S = \frac{0.2I}{x} \Delta p \Delta x$$

$$dS = \frac{0.2I \Delta p}{x} dx \quad (4)$$

Then the total stresses enclosed by the three surfaces are

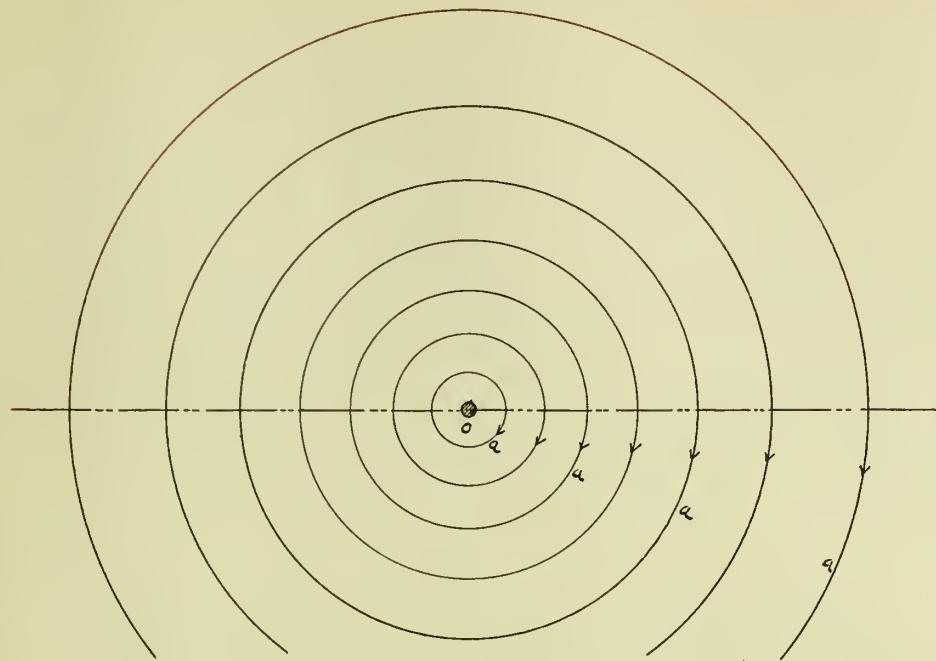


FIGURE 10.

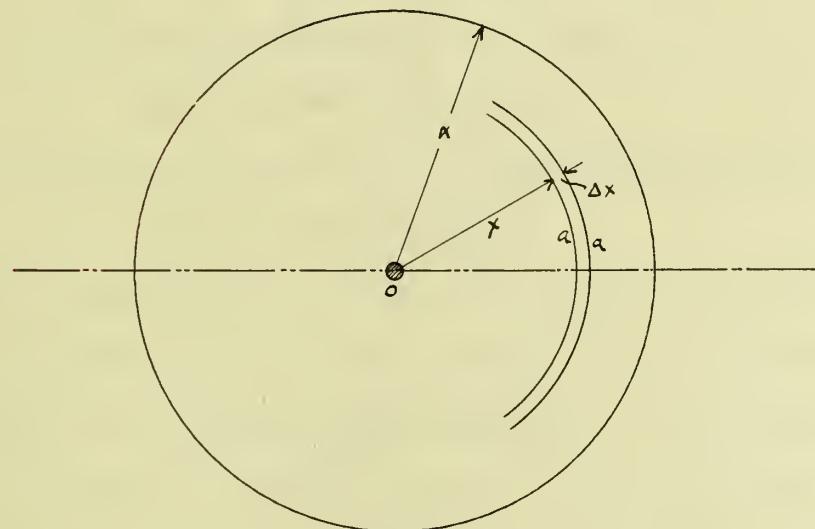


FIGURE 11.

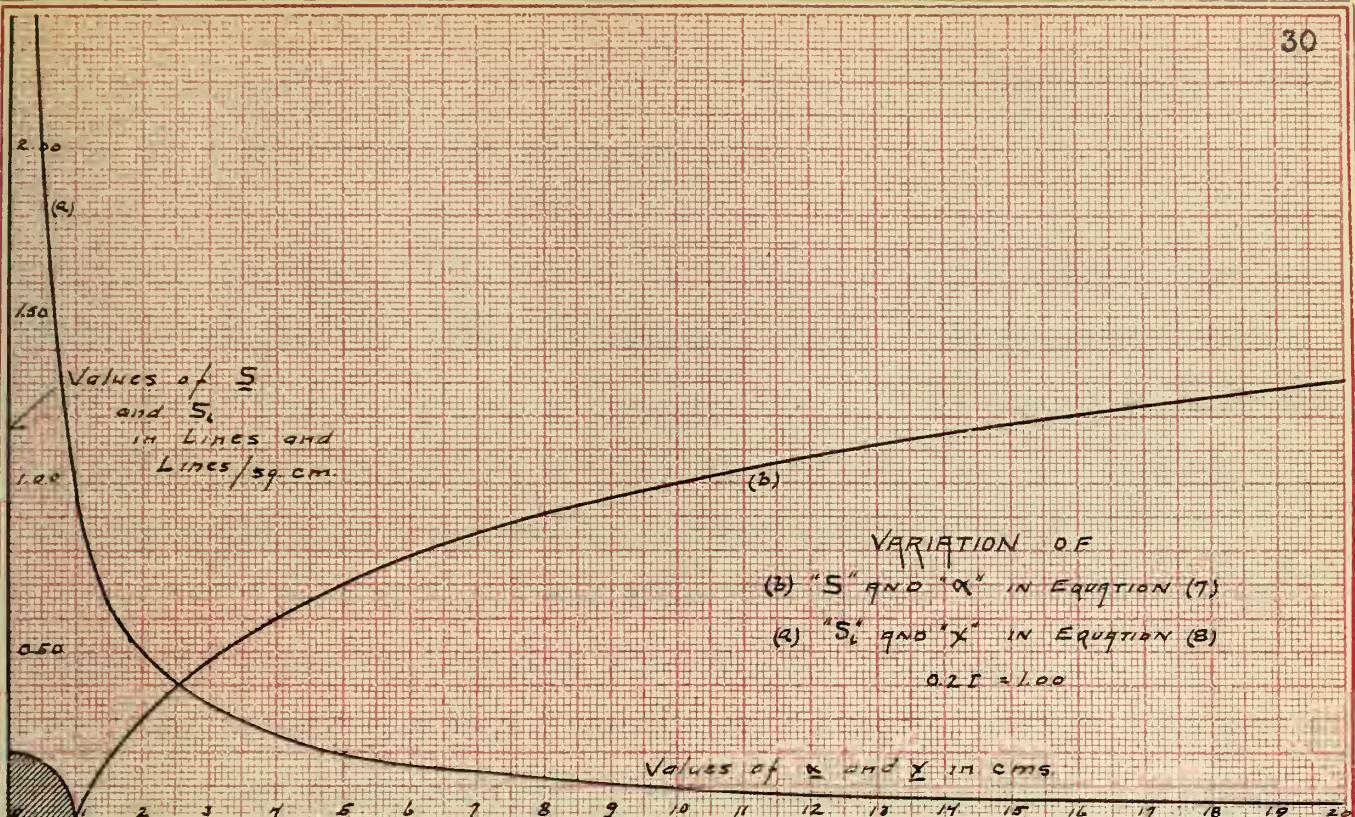


FIGURE 12.

$$S = 0.2 I \Delta \rho / \rho \int_{\rho}^{\alpha} \frac{dx}{x} \quad (5)$$

in which ρ is the radius of the wire. Then

$$S = 0.2 I \Delta \rho / \rho \log \frac{\alpha}{\rho} + C \quad (6)$$

Since $S=0$ when $\alpha=\rho$, the constant of integration becomes zero, and

$$S = 0.2 I \Delta \rho / \rho \log \frac{\alpha}{\rho} \quad (7)$$

If α is varied, equation (7) gives the total magnitude of the stresses, for any value of α , which are set up in the part of the medium enclosed by two planes, $\Delta \rho$ centimeters apart, and the cylindrical surface, the radius of which is α . Figure 12 shows the graph of this equation. It is to be noted that the zero point of S occurs at the point $\alpha=\rho$, due to the fact that S represents the flux outside the wire. The intensity at any point distant x centimeters from the surface of the wire is given by equation (3); and the stress in any annular ring distant x centimeters from the surface of the wire, by equation (4). Since the intensity may be used as the meas-

ure of the stress at a point, the relation of the stress to the distance from the center of the wire, is

$$x S_i = 0.2 I \quad (8)$$

provided that $x \leq \rho$, the radius of the wire. This relation also, is shown in Figure 12. Such, in general, is the quantitative distribution of the magnetic stresses outside of the wire carrying an electric current. Such determination holds for any value of the current I ; and the magnitude of the stress effects varies with I .

Next will be considered the magnetic stresses set up inside the conductor carrying the current. Suppose Figure 14, to represent an enlarged right section of the wire. Assume that the current I , flowing in the wire is equally distributed, that is, that the current density in the wire is the same in all parts of the right section. Let the radius of the section be ρ , and assume that the section shown is in one of two parallel planes, distant $\Delta\beta$ centimeters from each other. Consider first a circular element, having a radius x , and concentric with the surface of the wire. Then the current flowing in this element is

$$i = \frac{x^2}{\rho^2} I \quad (9)$$

The intensity of stress at any point distant β centimeters from the edge of the "x element", is

$$S_i = \frac{0.4 \pi x^2 I}{2 \pi \rho^2 (x + \beta)} \quad (10)$$

$$S_i = \frac{0.2 I x^2}{\rho^2 (x + \beta)}$$

Consequently the stress in an annular ring of right section $\Delta\rho\Delta\beta$ is

$$\Delta S = \frac{0.2 I x^2 \Delta\rho \Delta\beta}{\rho^2 (x + \beta)} \quad (11)$$

and the total stress in the section outside the "x element" is

$$S_i = \frac{0.2 I \Delta\beta}{\rho^2} \int_0^{\rho} \frac{x^2}{x + \beta} d\beta \quad (12)$$

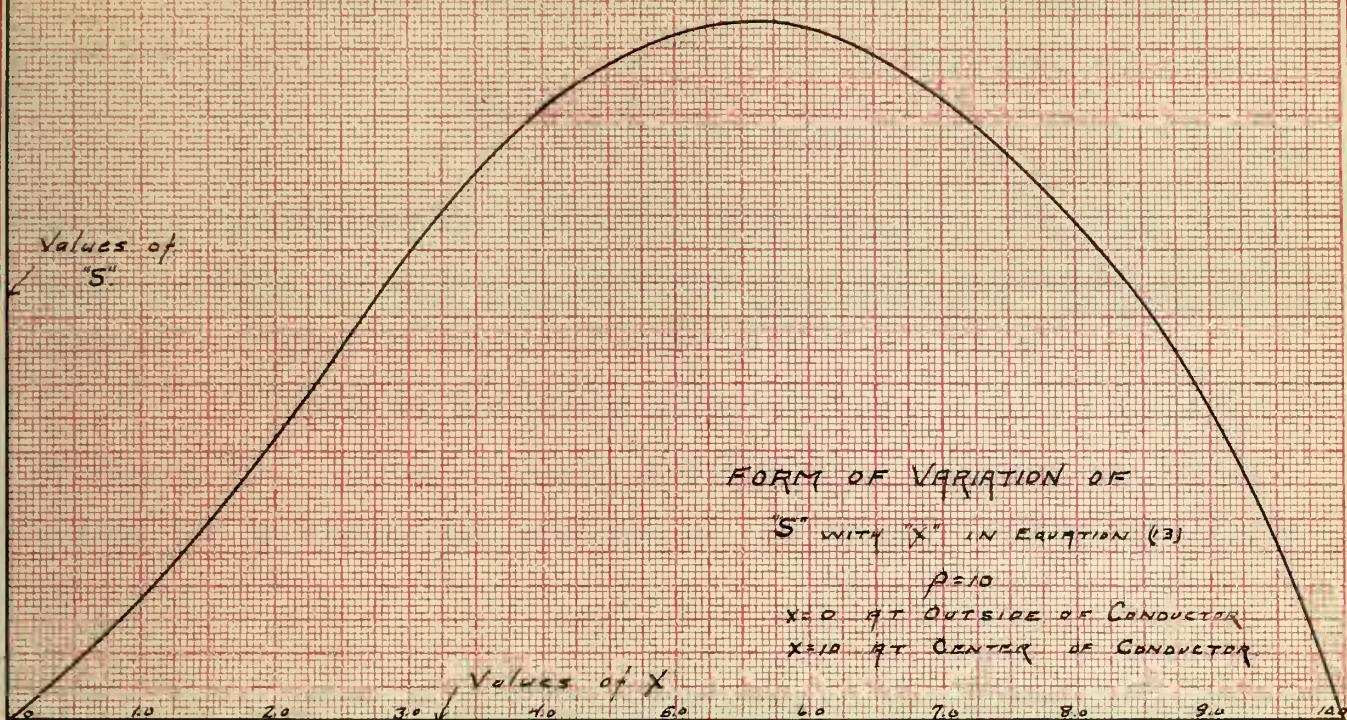


FIGURE 13.

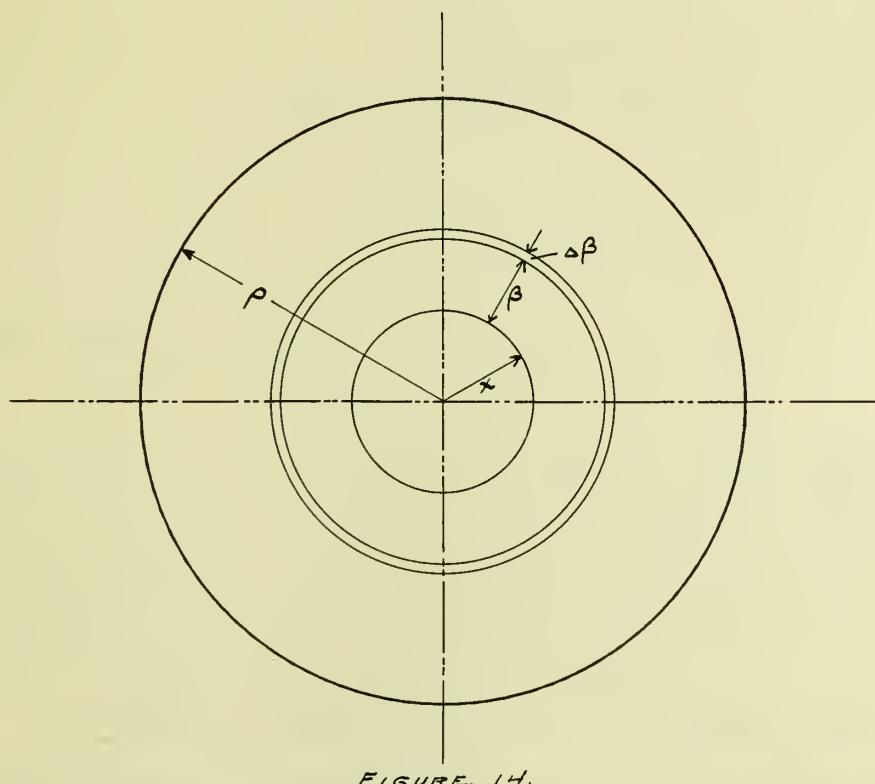


FIGURE 14.

Equation (12) integrates directly as a logarithmic function

$$S_r = \frac{0.2 I \Delta \beta}{\rho} x^2 \log \frac{x+\rho}{2x} \quad (13)$$

As x varies in equation (13), the value of S_r is the value of the total magnetic stresses set up inside the wire, and outside the " x element". Figure 13 shows the form of this variation.

Consider again the section of the wire in Figure 15, as in Figure 14. A total current I flows thru the section, and the stress set up inside the wire due to the current in the " x element", is

$$\iota = \frac{x^2}{\rho^2} I$$

Then for any value of x , the stress set up in an increment section $\Delta x \Delta \beta$, just outside the " x element", is

$$S_r = \frac{0.2 I \Delta \beta \Delta x}{\rho^2}$$

$$S_{TOTAL} = \frac{0.2 I \Delta \beta}{\rho^2} \int_0^{\rho} x dx$$

$$= 0.1 I \Delta \beta \quad (15)$$

and for unit length of wire

$$S_{TOTAL} = 0.1 I \quad (15a)$$

It is to be noted that, in equations (15) and (15a), S_r is a constant. Hence, for any size wire, having a given current I flowing therein, the total stress set up inside the wire is the same and is equal to the number of C.G.S. units of current flowing.

It follows, from the above development, that the variation of stress intensity inside the wire, with the distance from the center, is rectilinear. This may be shown in the following manner: Suppose Figure 16 to represent the right section of a conductor of

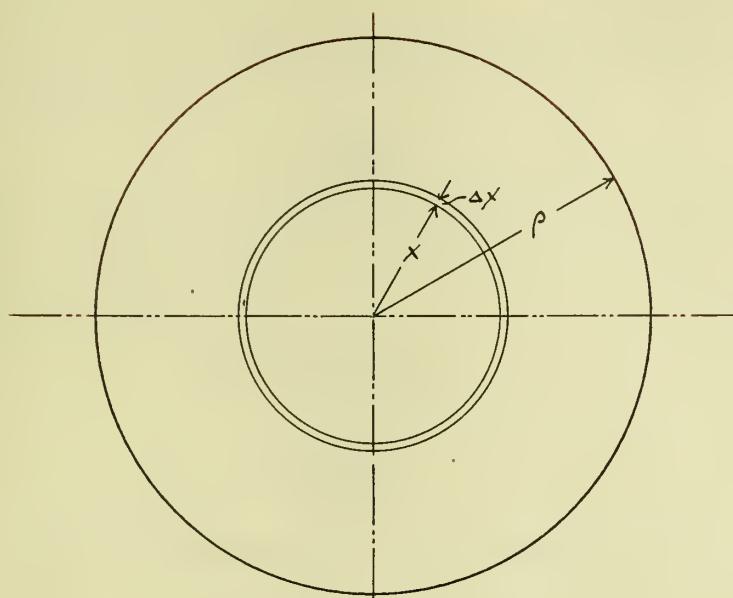


FIGURE 15

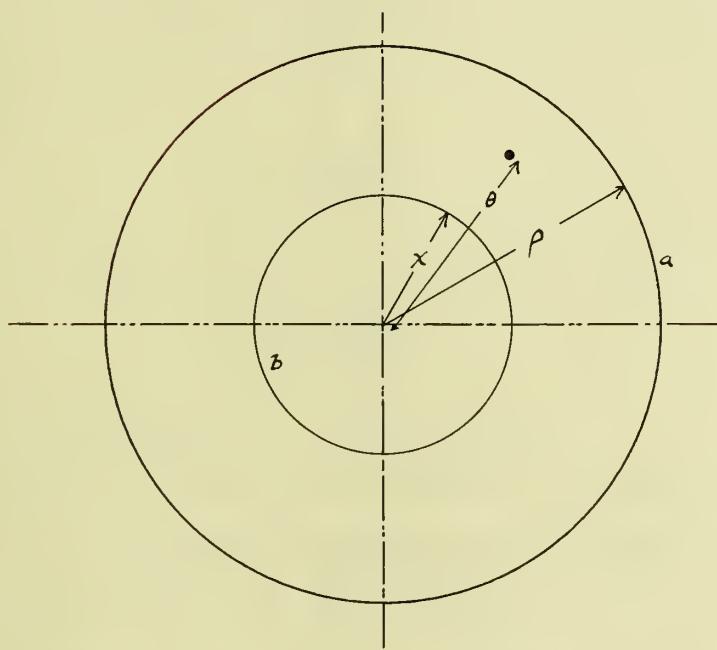


FIGURE 16

radius ρ . Then if a current I flowing in the conductor, is so distributed that equal current density is obtained in all parts of the section considered, the current within the concentrically circular part b , of radius x , is

$$I = \frac{x^2}{\rho^2} I$$

Then the stress intensity at any point at a distance θ from the center, due to this current, is

$$S' = \frac{0.4 \pi x^2 I}{2 \pi \theta \rho^2}$$

$$= \frac{0.2 x^2 I}{\rho^2 \theta}$$

Then, since, inside a conductor, the intensity of the stress at a point due to the sum of all the magnetic effects of the current intensities within a concentric circle thru this point, is equal to the intensity of the stress, which may be considered as due to the total amount of current flowing within a circle thru this point, x may be replaced by θ , and

$$S' = \frac{0.2 \theta I}{\rho^2} \quad (15b)$$

which is rectilinear in S' and θ . Also if $\theta = \rho$,

$$S' = \frac{0.2 I}{\rho}$$

which agrees with equation (8) when $x = \rho$.

Figure 17 shows the complete range of stress variation from the center of the wire outward, using stress intensity at any abscissa value as ordinate at that point.

Consider now, an arrangement of a certain number of wires all lying parallel, in the same plane, in a medium, spaced equidistantly, and each carrying the constant current I . A right section of such an arrangement is shown in Figure 19. Assume that the current flows in the same direction in each wire. Then, due to the

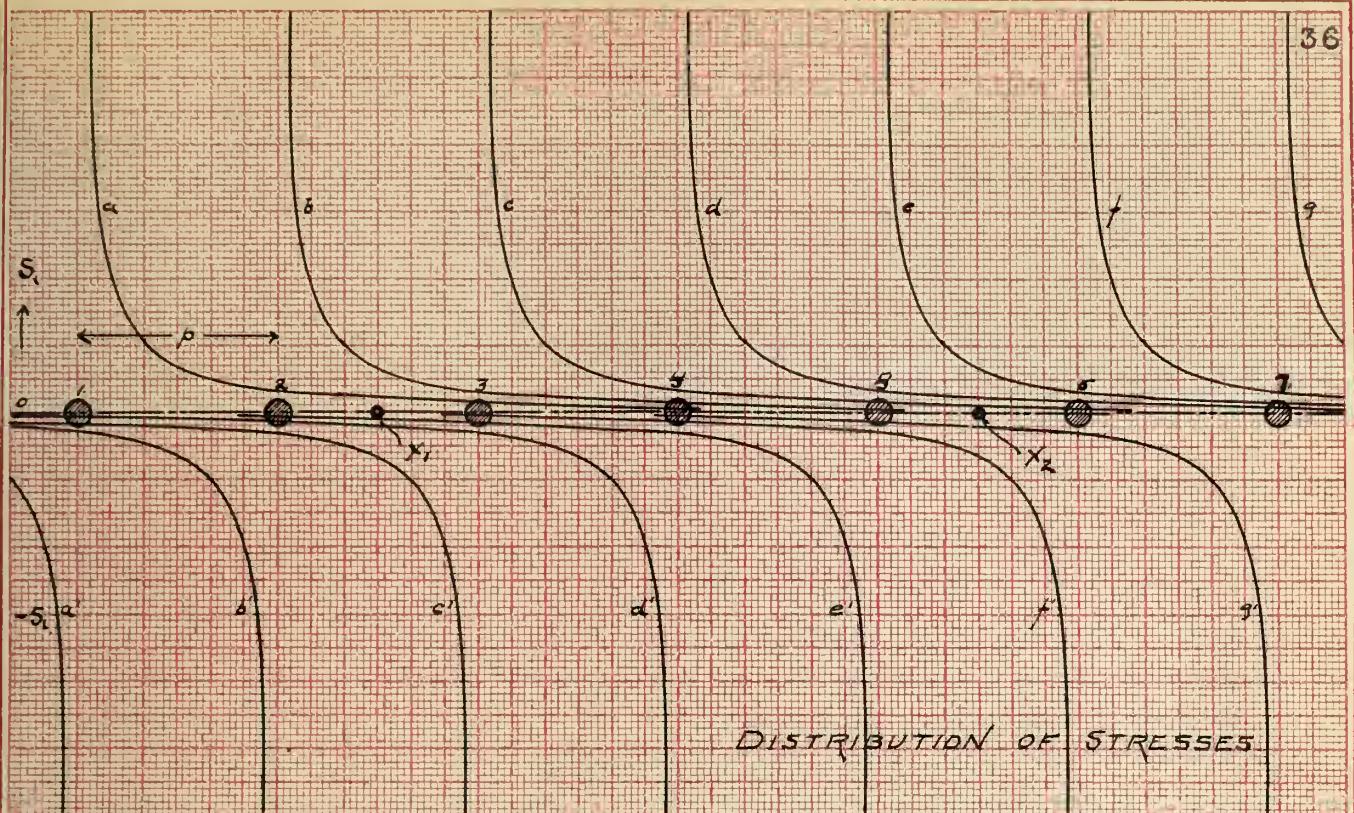


FIGURE 18.

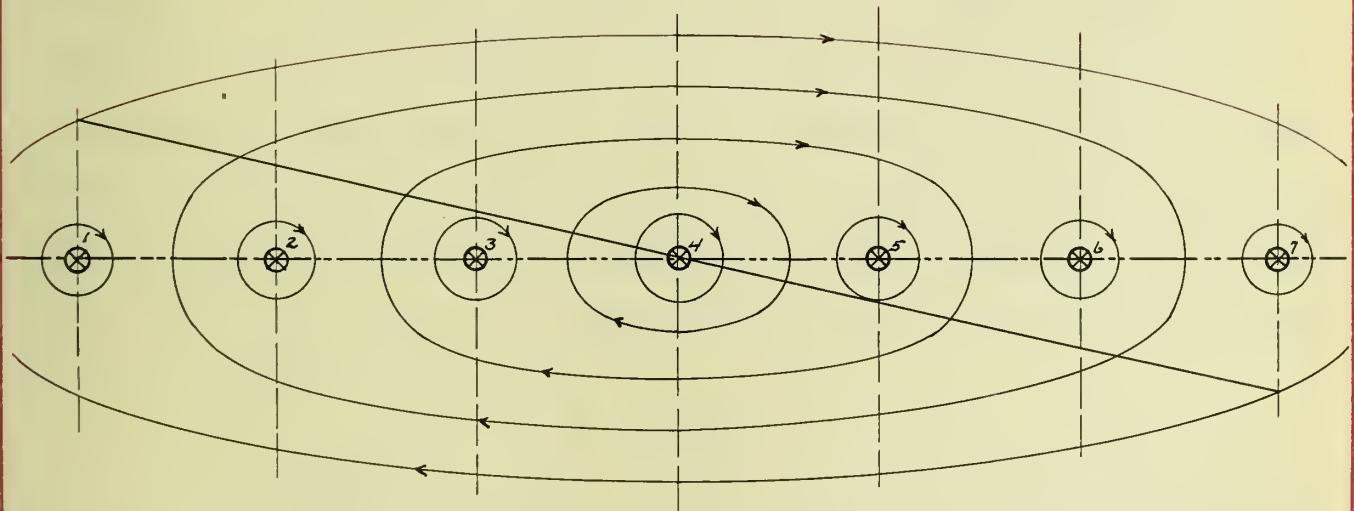


FIGURE 19.

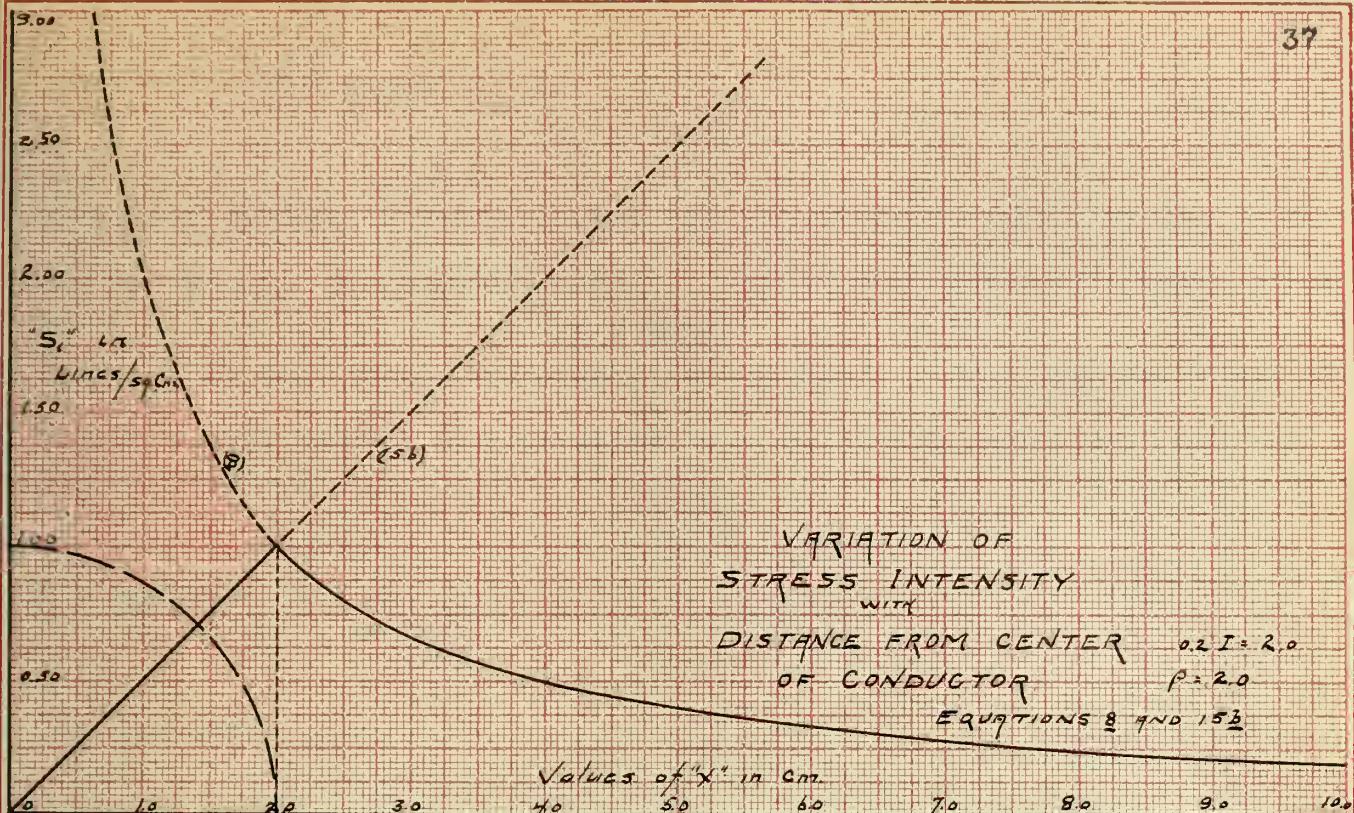


FIGURE 17.

magnetic effect of the current, there is a tendency for a field of magnetic stress to be set up in the medium and surrounding each wire separately, having the form shown in Figure 2. Since however, the lines of force, or the tubes of unit stress, cannot cross each other, the lines of stress resulting from the tendencies to produce a separate field for each wire, will have the general form shown in Figure 19. This can be shown, both mathematically and graphically. In equation (8), it was shown that the stress at any point distant x units from the center of a wire, is

$$S_i = \frac{0.2 I}{x}$$

Consider that the intersection of the coordinate axes occurs at the center of the wire #1; then at any point x, on the "x-axis", the stress intensity due to any one of the conductors considered, will have a value corresponding to this equation. Then for wire #1, a stress curve may be plotted as shown in Figure 18, a, a'; and the

equation of the curve a, is

$$S_c = \frac{0.2 I}{x}$$

In a like manner, a stress curve may be drawn for the wire #2, and the curves b, b', are the result. The equation for the curve b, is

$$S_c = \frac{0.2 I}{x-p}$$

where p is the distance between the centers of two adjacent conductors. In a like manner, the equation for the stress curve for wire #5, is

$$S_c = \frac{0.2 I}{x-4p}$$

The equation for the curve a', is directly,

$$S_c = \frac{-0.2 I}{(-x)}$$

for the curve b',

$$S_c = \frac{-0.2 I}{p-x}$$

and, for the curve e',

$$S_c = \frac{-0.2 I}{4p-x}$$

and correspondingly, for any one of the stress curves shown.

Now consider a point x, on the line of abscissae. It is desired to know the stress set up at this point, by the combined magnetic effect of the currents in the several conductors. For convenience, let x=1.5p, then the stress tendency at x, due to the current in conductor #1, is

$$S'_{x_1} = \frac{0.2 I}{1.5p}$$

The stress tendency due to the current in conductor #2, is

$$S''_{x_2} = \frac{0.2 I}{0.5p}$$

And that due to the current in conductor #3 is

$$S'''_{x_3} = \frac{-0.2 I}{0.5p}$$

In a similar manner, the stress tendencies due to the currents in the conductors #4, 5, 6, and 7, are respectfully

$$S''''_{x_4} = \frac{-0.2 I}{1.5p}$$

$$S_{x_1}'' = \frac{-0.2 I}{2.5 p}$$

$$S_{x_1}''' = \frac{-0.2 I}{3.5 p}$$

and

$$S_{x_1}'''' = \frac{-0.2 I}{4.5 p}$$

then the actual resultant stress at the point x_1 , is

$$S_{x_1} = \frac{0.2 I}{p} \left[\frac{1}{1.5} + \frac{1}{0.5} - \frac{1}{0.5} - \frac{1}{1.5} - \frac{1}{2.5} - \frac{1}{3.5} - \frac{1}{4.5} \right]$$

$$S_{x_1} = \frac{-0.182 I}{p} \quad (16)$$

Consider some other point x_2 , on this line ox , Figure 18. By the same process as before, the stress tendencies may be added, to give the resultant stress set up at that point. If, for convenience, $x_2 = 4.5 p$, the value of the stress will be

$$S_{x_2} = \frac{+0.182 I}{p} \quad (17)$$

If values of x , are taken representing points at the centers of the distances between the pairs of adjacent wires, as $x = .5 p$, $1.5 p$, $2.5 p$, and so forth, and the resultant stresses at these points are found, a curve may be plotted showing the resultant relation of x and s . It is evident, from an examination of Figure 18, that at the center of the distance between the two end conductors, at $x = 3 p$,

$$S_r = 0$$

Now, in Figure 20, the stress tendencies are plotted for each point midway between any two adjacent wires, and a curve is drawn thru the ends of the resulting ordinates. There is a positive, and a negative, curve, corresponding to the positive and negative resultant stress tendency for each point. The total resultant curve is then drawn, and is seen to be approximately a straight line. The separately

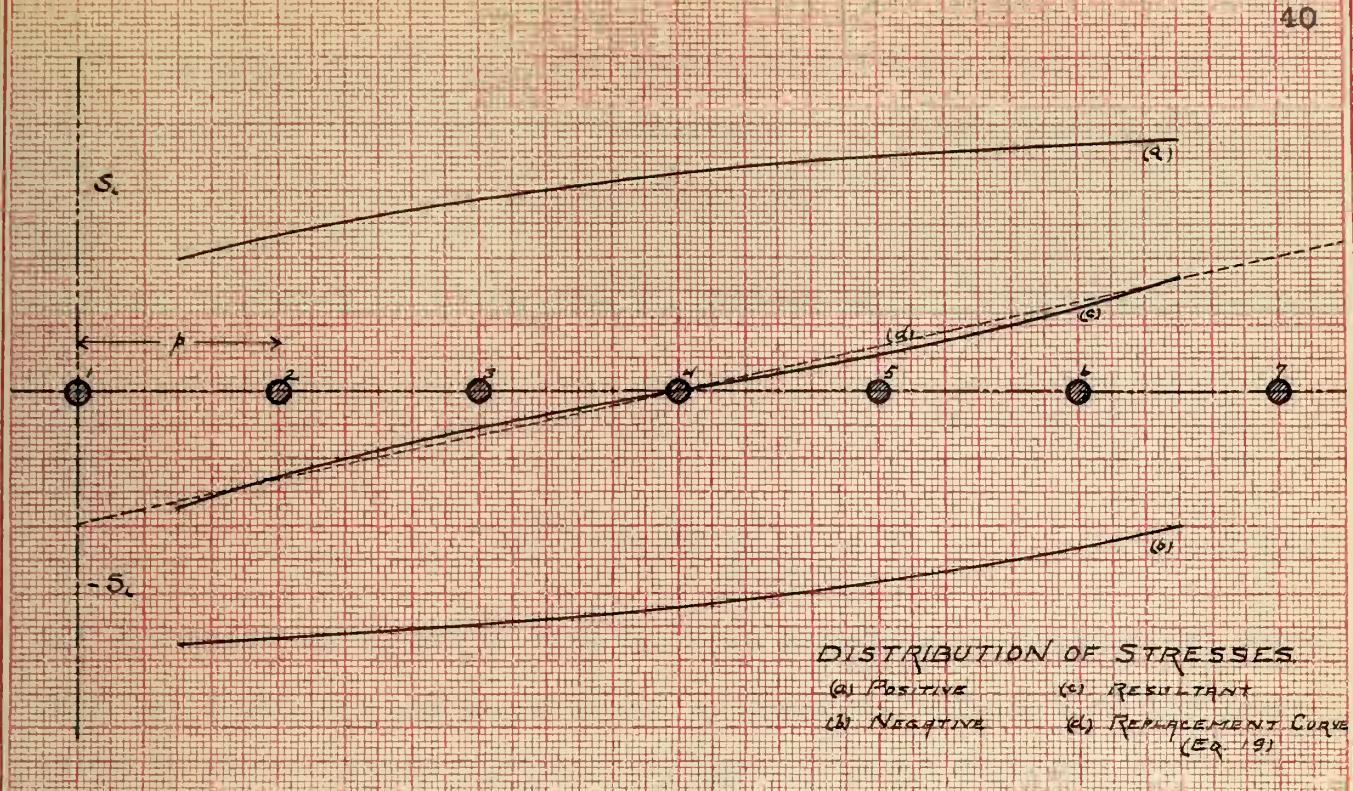


FIGURE 20.

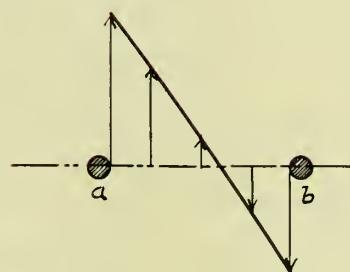


FIGURE 21.

drawn tendency curves are very nearly logarithmic, and they may be considered to represent the equations

$$S_1 = K \log x + K' \quad (18a)$$

and

$$S_2 = K \log (b\rho - x) + K' \quad (18b)$$

If these curves (18a and 18b) are subtracted, one from the other, the resultant curve is

$$S_r = -K \log \frac{x}{b\rho - x} \quad (19)$$

and the curve of equation (19) is that shown at d, Figure 20, drawn thru the resultant curve obtained by actually subtracting the ordinates of the tendency curves. It must be noted that the curve d, Figure 20, holds only for the space between the outside wires, and near these conductors gives values for the resultant stress, which are smaller than the actual values. Also, the resultant curve must be considered as the resultant or mean of all the separate resultant curves which might be drawn for any of an infinite number of other series of values for x, having the general form of

$$x = \frac{p}{n}, \frac{p}{n} + p, \frac{p}{n} + 2p, \frac{p}{n} + 3p, \frac{p}{n} + 4p, \frac{p}{n} + 5p$$

Now, it is evident, that in order that the stresses may follow the law just derived, the general form of the field of stress in a plane perpendicular to the wires, must be that shown in Figure 19. Outside the end conductors, since there are no opposing stresses, the stress tendencies are so combined that the law of rectilinear variation does not hold.

Equation (19) may, for all general purposes, be replaced by

$$S_r = \frac{1.32 I}{\rho^2} x - 3.96 \frac{I}{\rho} \quad (20)$$

Equation (20) does not give accurate results in the vicinity of the end conductors of the set considered; but it is sufficiently accurate to obtain general quantitative values for the stresses set up at any one point along the line OX, provided that this point is chosen near the center of the space between any two adjacent conductors. This proviso is necessary since the stress set up at any point between conductors which are separated by a distance much larger than the diameter of the conductors themselves, is represented by the distribution curve shown in Figure 21. The conductors a b in the figure are assumed to be two of a number of conductors arranged as in Figure 19. It is not necessary to investigate this distribution, since there will be no application for it.

The phenomena considered above mathematically, may be investigated graphically by a method illustrated in Figure 22. The conductors numbered from "1" to "5", are shown and a current is assumed to be flowing in each conductor in a direction into or toward the paper. A number of points, o, o, o, are then chosen, located symmetrically about the five conductors, and all lying in the plane under consideration. Now consider any point q. Assuming some numerical value representing a given stress set up at unit distance from one conductor, the stress at any distance may be found by means of equation (8)

$$\times S_c = K$$

Also, it is known that the intensity at any point due to a single conductor, is in a direction perpendicular to a line from the point to the center of the conductor. Then, knowing the intensity and direction of the stress tendency at any point in the field of the five wires, Figure 22, vectors may be drawn, representing the stress tendencies of all the wires at the point under consideration. Thus,

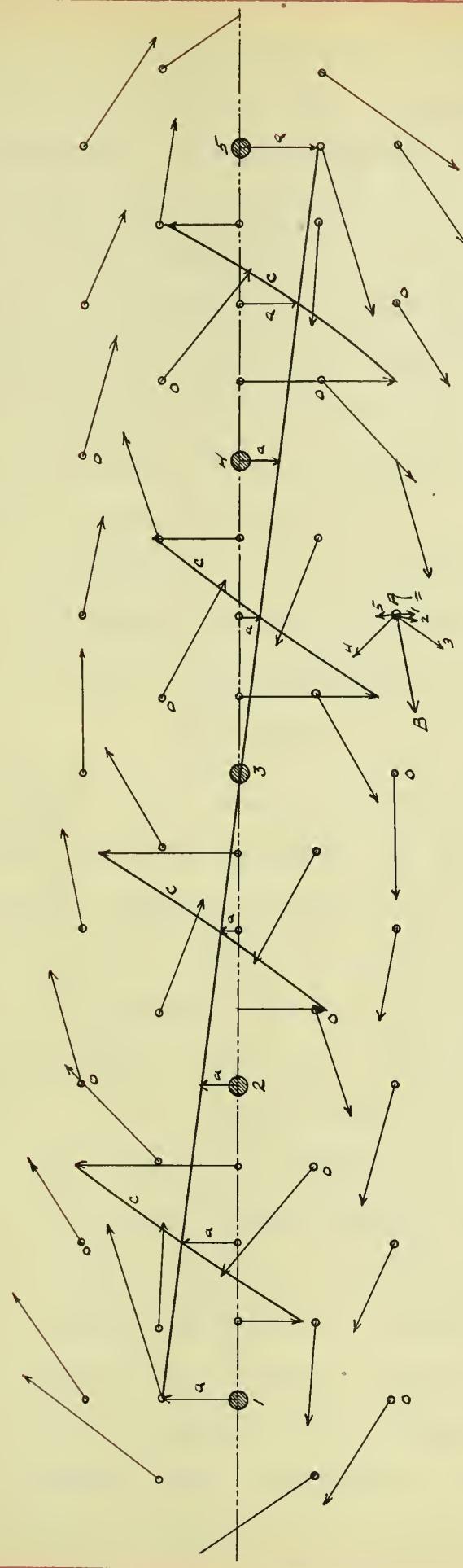


FIGURE 22.

vectors, 1, 2, 3, 4, 5, at 7, Figure 22, correspond to conductors 1, 2, 3, 4, 5. Then the actual stress intensity at the point 7, due to the five conductors, is represented by the vector 7G. Such construction may be made for any other points, o, o, and the result of a large number of these constructions is the general form of the field, as is shown in the figure. An investigation of this figure reveals the fact that the lines of stress have in general, the arrangement of those in Figure 19. In Figure 22, the line ps, corresponds to the graph of equation (20); and the lines, c, c, are the curves corresponding to that in Figure 21.

Such in general, is the form of the field around conductors arranged as in Figure 19, having a space between adjacent conductors, a number of times larger than the diameter of the conductor, or

$$\rho = 2N\rho$$

where N is larger than 3 or 4. Consider now the condition in which N is between 1.1 and 2.0, that is where the wires are so close together that the distance between the surfaces of adjacent wires is equal to, or less than, the diameter of one of them. Such an arrangement is shown in Figure 23. The lines ab, ab, are the curves of the stress intensity within the wires, corresponding to equation (15a). The lines c, c, c, are the curves for the stress intensity outside the wires, corresponding to equation (8). At any point x, along the line MN, the actual stress intensity may be found by the summation of all the stress tendencies at the point, due to the current flowing in the several conductors. The resultant curve obtained by this method, is shown by the irregular curve in Figure 23. It is seen that each way from the center of the group, the intensity rises. Inside the center conductor there are lines of stress which are practically concentric with the center of the conductor. In no other one

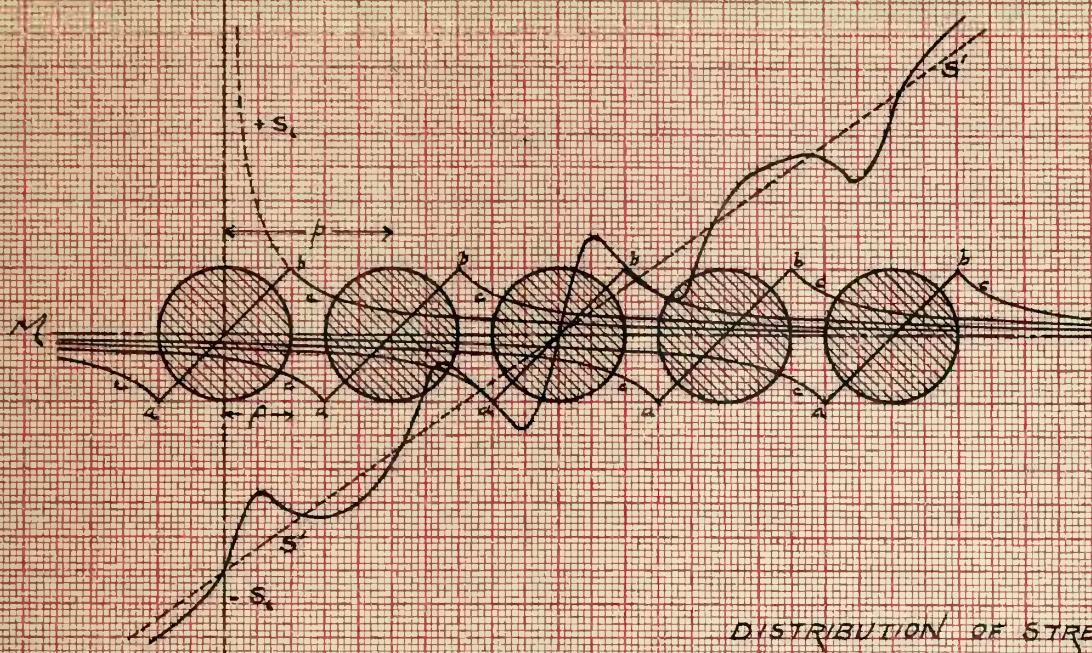


FIGURE 2.3.

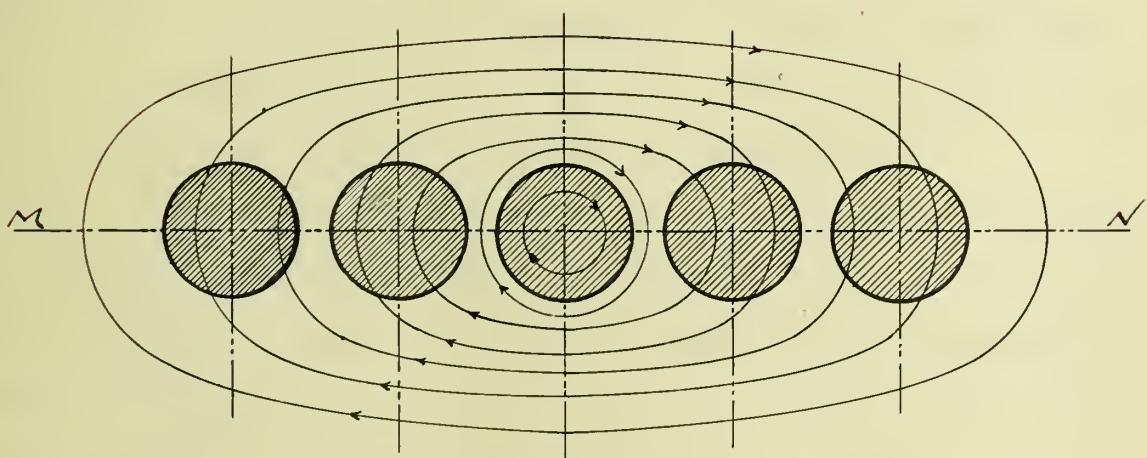


FIGURE 2.4.

of the conductors are such lines to be found; but there are lines passing thru the several conductors and around the center one. The general form of the field is shown in Figure 24. The effect of placing the wires an extremely small distance apart, is simply to crowd the curve (Figure 23) together, from the ends of the group toward the center. It is evident that a straight line s, may be drawn thru the irregular curve in Figure 23, and that this line may be used to determine the stress intensity at any point along the line MN. This is a valuable condition, since an exact analytical expression for the irregular curve cannot be found without the utilization of very complicated mathematics. It is possible to make a closer approximation by the use of equation (19)

$$S = -K \log \frac{x}{(n-1)(p-x)} \quad (20a)$$

where n is the number of wires, p is the distance between the centers of adjacent wires, and x must be considered in units of p, as x = 0.6p, 3p, et cetera. Figure 24 gives the general form of the resultant field, around the five conductors considered; and Figure 25 shows the distribution of stress in the field indicated in Figure 23, but also extending outside the group of conductors.

Now consider the number of wires to be increased and that they are arranged as shown in Figure 26. If again, the current flows in the same direction in all the wires and the magnetic action of the current is considered, it is evident that a stress curve will be obtained such as may be represented by the line ab. This is due to the fact that the stress tendencies in this case, follow the same general behavior as has been considered. Then the resultant field of stresses has the general form shown by the lines c, c, c. Also the ordinate of the curve at the point T, may be found by adding

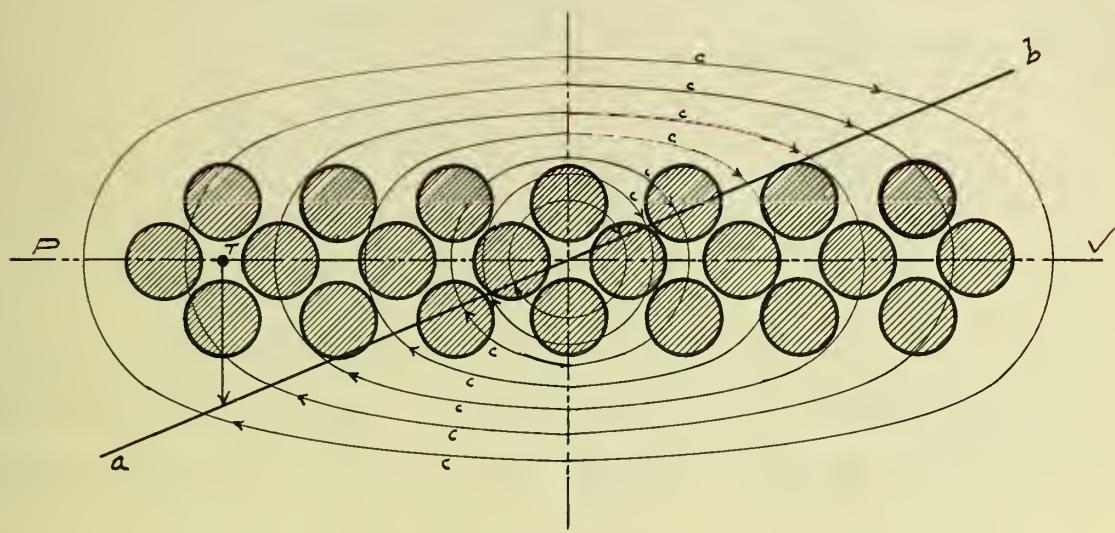


FIGURE 26.

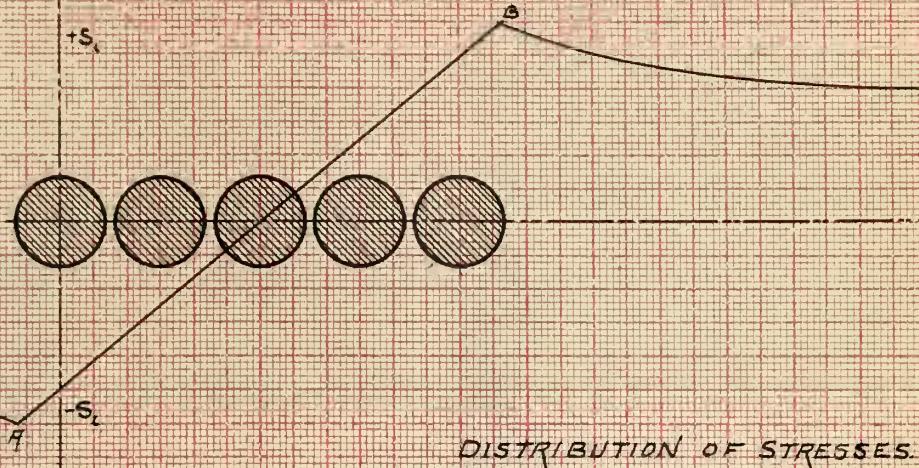


FIGURE 25

the stress tendencies of all the wires at that point. It is evident that vector addition will be necessary and the expression for the stress tendency due to each wire off the line PV, will be of the form

$$S = \frac{o \cdot z I}{x} \cos \phi$$

If a single conductor is considered to be wound into a cylindrical coil of a single layer, and a number of turns, the field of stresses for one side of the coil will, in general, have the form above considered. Such an arrangement is shown in Figure 27, which represents a section thru the axis AB, of the coil. In the center of the coil, the lines of stress have all the same general direction and outside the coil also, this characteristic holds. As before, the lines ab, and a'b, show the resultant stresses set up along the lines EF, and E'F'. Consider now, the point o, at the center of the coil. If the current I flows thru the coil, the turn #5,

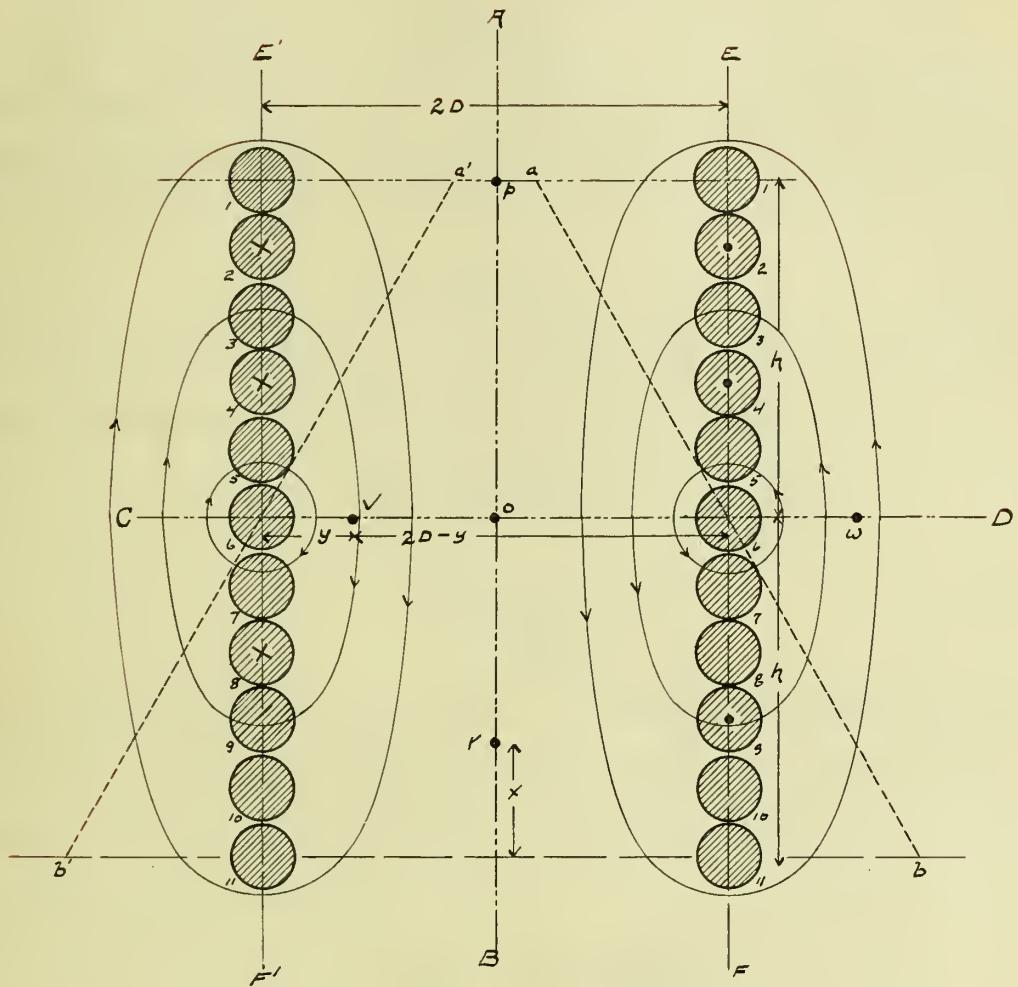


FIGURE 27.

has a certain stress tendency at the point, in the direction AB.

This tendency may be expressed as follows: The total magnetomotive force due to turn #5 is

$$F = 0.4\pi I \times 2\pi D$$

$$= 0.8\pi^2 ID \quad (21)$$

Then since the point o, is at a distance D, from the turn, under consideration, the stress intensity at the point due to this single turn is

$$S_1 = 0.4\pi I \quad (22)$$

Consider any turn, distant x units from the center turn of the coil.

The stress intensity at the point o due to this turn, is

$$S_1 = \frac{0.8\pi^2 ID}{2\pi(x^2+D^2)^{\frac{1}{2}}} \times \frac{D}{(x^2+D^2)^{\frac{1}{2}}}$$

$$= \frac{0.4\pi ID^2}{x^2+D^2}$$

Consider a portion dx, of the coil, having unit length, and containing n turns; let this portion of the coil be at a distance x from the center turn of the coil. Then the stress tendency at o due to this portion of the coil is

$$S_2 = \frac{0.4\pi n ID^2}{x^2+D^2}$$

If z is the length of the coil, then the total stress at the point o, due to the whole coil, is

$$S_o = 2 \int_0^z \frac{0.4\pi n ID^2}{x^2+D^2} dx \quad (23)$$

Then

$$S_o = 0.8\pi n ID \arctan \frac{z}{D} \quad (23a)$$

and S_o has the direction AB.

If the coil has unit radius,

$$S_0 = 0.8\pi n I \arctan h \quad (23b)$$

$$= \frac{0.4\pi NI}{h} \arctan h$$

Suppose the point to be located at \underline{p} , on the line \underline{AB} , and in the plane of turn #1. Then the total stress set up by the N turns on the coil, is

$$S_p = \int_0^{2h} \frac{0.4\pi NI D^2}{2h(x^2 + D^2)} dx$$

and

$$S_p = \frac{0.4\pi NI D}{2h} \arctan \frac{2h}{D} \quad (24)$$

If \underline{r} represents any point in the line \underline{AB} , between turn #1 and turn #11, the stress at \underline{r} , due to the turns below \underline{r} is

$$S_r = \int \frac{0.4\pi I n D^2}{x^2 + D^2} dx$$

$$= \frac{0.4\pi NI D}{2h} \arctan \frac{x}{D} \quad (25)$$

Also the stress at \underline{r} , due to the turns above \underline{r} , is

$$S_{r_1} = \int \frac{0.4\pi n I D^2}{(2h-x)^2 + D^2} dx$$

$$= \frac{0.4\pi NI D}{2h} \arctan \frac{2h-x}{D} \quad (26)$$

Then the total stress at any point \underline{r} in the line \underline{AB} , distant x units from the bottom of the coil, is

$$S_r = \frac{0.2\pi NI D}{h} \left[\arctan \frac{x}{D} + \arctan \frac{2h-x}{D} \right] \quad (27)$$

Figure 28 shows the graph of equation (27), and the ordinates represent the total stress intensity at any point \underline{r} in the line, \underline{AB} , the axis of the coil.

If \underline{v} is any point in the center line, \underline{CD} , of the sec-

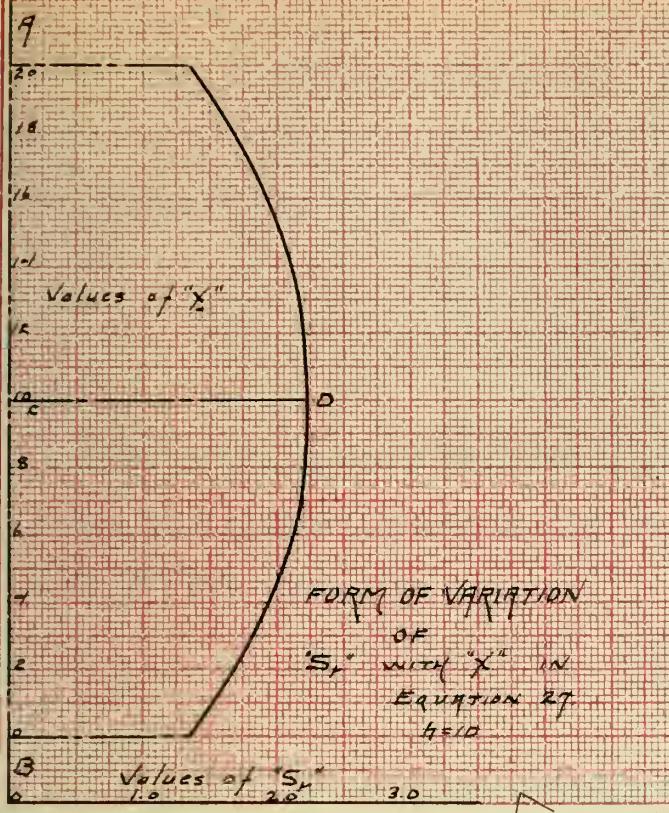


FIGURE 28

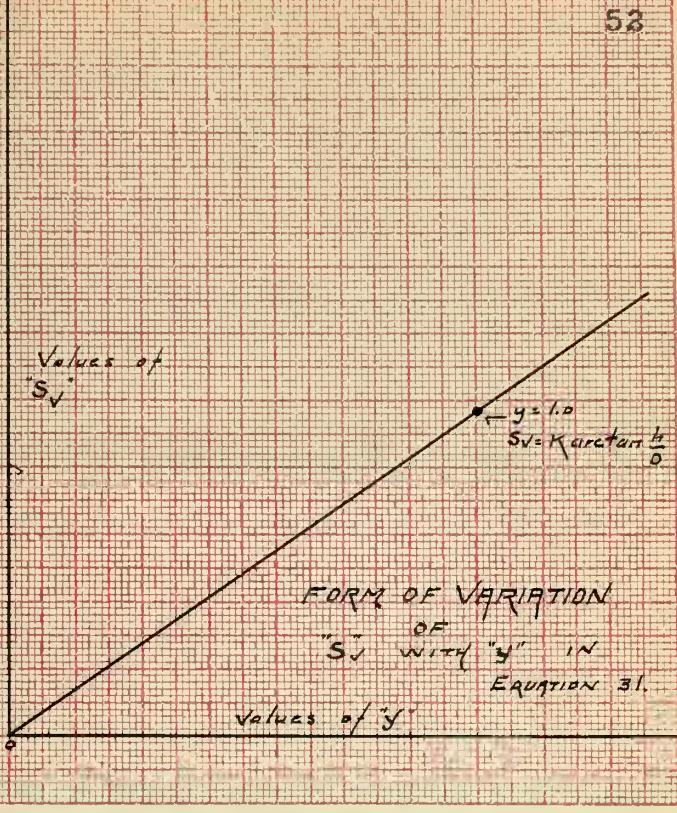


FIGURE 29

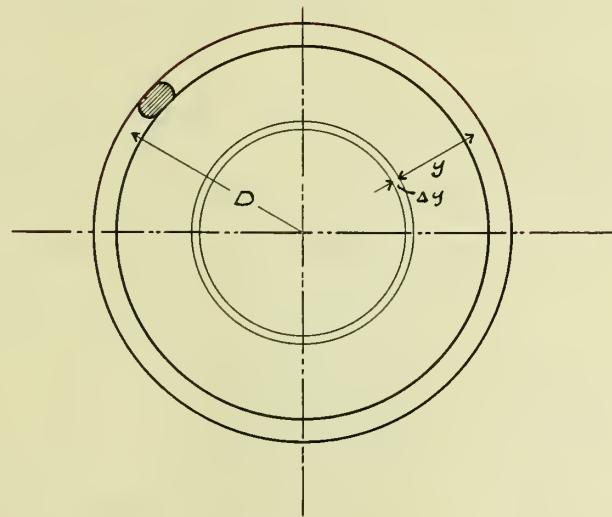


FIGURE 30.

tion shown in Figure 27, distant y units from the center of the left section of the coil, the stress intensity at \checkmark due to the sections of turns in the right side of the coil, is

$$S_v' = z \int_0^h \frac{0.4\pi n I (2D-y)}{2\pi((2D-y)^2 + x^2)} dx$$

$$S_v' = 0.4 n I \arctan \frac{h}{2D-y} \quad (28)$$

Also, the stress intensity due to the sections of turns in the left side of the coil, is

$$S_v'' = 2 \int_0^h \frac{0.4\pi n I y}{2\pi(y^2 + x^2)} dx$$

$$S_v'' = 0.4 n I \arctan \frac{h}{y} \quad (29)$$

Then the total stress intensity at \checkmark , due to the two sections of the coil, (these are assumed to have unit length), is

$$S_v''' = 0.4 n I \left[\arctan \frac{h}{2D-y} + \arctan \frac{h}{y} \right] \quad (30)$$

Since, as y increases in value from zero to D , the intensity at \checkmark becomes, more and more, the sum of the intensities due to the all the increments of length of the wires in the turn, and assuming this variation to be rectilinear,

$$S_v = 0.4\pi n I y \left[\arctan \frac{h}{2D-y} + \arctan \frac{h}{y} \right] \quad (31)$$

and S_v denotes the intensity of stress in a direction parallel to AB , at any point \checkmark , in the line CD , as y varies in value from zero to D . Figure 29 shows the graph of equation (31).

Suppose that Figure 30 represents a right section of the coil, made on the line CD . Let y vary from zero to D . The total magnetic stress set up in the annular ring of width dy , is the stress

intensity at y , multiplied by the area of the ring. Then

$$S_s' = 0.4\pi n I y \left[\arctan \frac{h}{2D-y} + \arctan \frac{h}{y} \right] \times 2\pi(D-y) dy$$

and

$$S_s = \int_0^D \frac{0.4\pi^2 N I y (D-y)}{h} \left[\arctan \frac{h}{2D-y} + \arctan \frac{h}{y} \right] dy \quad (32)$$

$$S_s = \int_0^D \frac{0.4\pi^2 N I y (D-y)}{h} \left[\arctan \frac{2Dh}{y(2D-y)+(-h^2)} \right] dy \quad (32a)$$

This equation may be integrated by the substitution of

$$\tan z = \frac{2Dh}{y(2D-y)-h^2}$$

Equation (32a) then becomes

$$S_s = \int z \left[D \pm (D^2 - h^2 - 2Dh \cot z)^{\frac{1}{2}} \right] 2Dh \csc^2 z dz$$

and

$$S_s = \int 2D^2 h \csc^2 z dz \pm \int 2Dh \left[D^2 - h^2 - 2Dh \cot z \right]^{\frac{3}{2}} z \csc^2 z dz$$

By the integration by parts

$$S_s = 2Dh \left[-z \cot z + \log \sin z \right] \pm \frac{2z}{3} \left[D^2 - h^2 - 2Dh \cot z \right]^{\frac{3}{2}} - \int \frac{2}{3} \left[D^2 - h^2 - 2Dh \cot z \right]^{\frac{3}{2}} dz \quad (33)$$

The integral in the second member of equation (33) is impossible of exact solution, although, by means of a slowly converging series, an approximation may be made, requiring a large number of terms for a reasonable degree of accuracy.

Substituting for the value of z , in equation (33), and placing the result between limits, as in equation (32)

$$S_s = (D^2 - h^2) \arctan \frac{2Dh}{D^2 - h^2} + 2Dh \log \sin \left[\arctan \frac{2Dh}{D^2 - h^2} \right] -$$

$$\frac{2}{3} \left(2h \right)^{\frac{3}{2}} \arctan \frac{2Dh}{D^2 - h^2} + h^2 \arctan \left(-\frac{2D}{h} \right) + 2Dh \log \sin \arctan \left(-\frac{2D}{h} \right)$$

$$= \frac{2}{3} (D^2 - 2h^2)^{\frac{3}{2}} \arctan \left(-\frac{2D}{h} \right) + \int_0^D \frac{2Dh(D^2 - 2h^2 - y(2D-y))^{\frac{3}{2}}}{(2Dh)^2 + (y(2D-y) - h^2)^2} dy$$

and the expression becomes extremely intricate.

Equation (33) however, can be expressed very much more simply by another method. Consider again, equation (31). S_V may be expressed as a function of the arctan z, where z is the sum of the angles in equation (31). That is

$$S_V = 0.4\pi n I y \arctan \frac{-2Dh}{y(2D-y) - h^2} \quad (34)$$

as in equation (32a). Now by plotting equation (34), it is seen that the resultant curve is rectilinear, and may be represented by the simpler equation,

$$S_V = 0.8\pi n I y \arctan \frac{h}{D} \quad (35)$$

The graph of this equation is shown in Figure 29.

Then the total stress set up across the section of the coil at the center, is

$$S_s = \int_0^D 0.8\pi n I \arctan \frac{h}{D} \times 2\pi y(D-y) dy$$

$$= \frac{1.6\pi^2 N I D^3}{12h} \arctan \frac{h}{D}$$

$$= 1.305 \frac{N I D^3}{h} \arctan \frac{h}{D} \quad (36)$$

and equation (36) gives the total number of unit lines of stress passing across the section at the center of the coil, or on the line CD, in Figure 27.

Now, it has been found that the stress intensity at the point P, Figure 27, at the center of turn #1, is (equation 24)

$$S_p = \frac{0.4 \pi N ID}{2h} \arctan \frac{2h}{D}$$

and that the stress intensity at the point o, in the center of the coil, (equation 23b), is

$$S_o = \frac{0.4 \pi N ID}{h} \arctan \frac{h}{D}$$

Then, the ratio of the former to the latter, is

$$\frac{S_p}{S_o} = \frac{\arctan \frac{2h}{D}}{2 \arctan \frac{h}{D}} \quad (37)$$

It is assumed that the distribution of stress intensity thruout the right section of the coil varies, as the section moves from the center to the top of the coil, in the same manner that the stress intensity at the center of the section, varies. This assumption is justified, due to the symmetrical arrangement of the turns upon the coil. Then the total stresses set up across a right section, thru turn #1, is

$$S_1 = \frac{1.6 \pi^2 N ID^3}{12h} \frac{S_p}{S_o} \arctan \frac{h}{D}$$

$$= \frac{1.6 \pi^2 N ID^3}{24h} \arctan \frac{2h}{D}$$

$$= 6525 \frac{N ID^3}{h} \arctan \frac{2h}{D} \quad (38)$$

Then the difference between the total stresses thru the section at CD, and those thru the section at turn #1, is

$$\begin{aligned} S_D &= 1.305 \frac{N ID^3}{h} \arctan \frac{h}{D} - 1.305 \frac{N ID^3}{2h} \arctan \frac{2h}{D} \\ &= 1.305 \frac{N ID^3}{h} \left[\arctan \frac{h}{D} - \frac{1}{2} \arctan \frac{2h}{D} \right] \end{aligned} \quad (39)$$

Then S_D is the total value of the stresses set up across the right

section at the center of the coil, which are not set up across the right section at the end of the coil. Therefore S_p is the total number of lines of unit stress which pass out thru the sides of the coil between the center and the ends, or, is the total flux leaking thru the sides of the coil.

Now, consider again, equation (27),

$$S_v = \frac{0.2 \pi NID}{h} \left[\arctan \frac{h}{D} + \arctan \frac{2h-x}{D} \right] \quad (39a)$$

the graph of which is shown in Figure 28. If the difference of any ordinate of this curve, from the maximum ordinate, is found, the expression of this difference is

$$S_v = \frac{0.4 \pi NID}{h} \arctan \frac{h}{D} - \frac{0.2 \pi NID}{h} \left[\arctan \frac{h}{D} + \arctan \frac{2h-x}{D} \right]$$

directly from equations (23a) and (27). Then

$$S_v = \frac{0.2 \pi NID}{h} \left[2 \arctan \frac{h}{D} - \arctan \frac{h}{D} - \arctan \frac{2h-x}{D} \right]$$

If the average ordinate S'_v , of S_v , is found, and the ratio $\frac{S'_v}{S_p}$, be multiplied by S_p , the result is the amount of leakage flux, or the number of lines of the leakage stresses, which may be considered to be interlinked with all the turns on the coil. Now

$$S'_v = \frac{1}{h} \int_0^h S_v dx$$

and after a series of transformations

$$S'_v = \frac{0.2 \pi NID}{h^2} \left[2h \left(\arctan \frac{h}{D} - \arctan \frac{2h}{D} \right) - \frac{D}{2} \log \left(1 + \left(\frac{2h}{D} \right)^2 \right) \right]$$

and from equation (27)

$$\frac{S'_v}{S_p} = \frac{\frac{0.2 \pi NID}{h^2} \left[2h \left(\arctan \frac{h}{D} - \arctan \frac{2h}{D} \right) - \frac{D}{2} \log \left(1 + \left(\frac{2h}{D} \right)^2 \right) \right]}{\frac{0.2 \pi NID}{h} \frac{1}{2} \left[\arctan \frac{h}{D} - \frac{D}{2} \arctan \frac{2h}{D} \right]}$$

and, multiplying by S_p , equation (39)

$$S_L = \frac{5.23 NID^3}{h} \left[\arctan \frac{h}{D} - \arctan \frac{2h}{D} - \frac{D}{4h} \log \left(1 + \left(\frac{2h}{D} \right)^2 \right) \right] \quad (40)$$

Now, if \underline{S} is multiplied by $\frac{\mathcal{N}}{I}$, the result is the total number of interlinkages of the leakage lines of unit stress with turns of the coil, per unit current; or, is the leakage inductance of the coil, and may be denoted by \underline{L}' . Then

$$\underline{L}' = \frac{5.23 \mathcal{N}^2 D^3}{h} \left[\arctan \frac{h}{D} - \arctan \frac{2h}{D} - \frac{D}{4h} \log \left(1 + \left(\frac{2h}{D} \right)^2 \right) \right] \quad (41)$$

and \underline{L}' is the leakage inductance of the coil.

Now, all the lines of unit stress which pass across the section thru the turn #1, interlink with the whole number of turns. then the inductance due to these lines, is

$$\underline{L}'' = \frac{.6525 \mathcal{N}^2 D^3}{h} \arctan \frac{2h}{D} \quad (42)$$

and the total inductance of the coil is

$$\underline{L} = - \frac{\mathcal{N}^2 D^3}{h} \left[5.23 \arctan \frac{h}{D} - 4.58 \arctan \frac{2h}{D} - \frac{D}{4h} \log \left(1 + \left(\frac{2h}{D} \right)^2 \right) \right] \quad (43)$$

This formula is not exact; but it is believed to be an approximation which will give results as accurately as they can be determined by means of the instruments in general use in the laboratory. Its use is limited to cylindrical coils, in which \underline{h} has a value equal to, at least, $2D$; also, the thickness of the coil must not be excessive, that is, the ratio of the mean radius of the coil to the radius of the inside layer of turns, must not be greater than approximately, 1.2. If a number of layers greater than unity is employed, \underline{D} becomes the mean radius of the coil. From the derivation of \underline{L}' however, it is evident that provided \underline{D} denotes the mean radius of the coil, the value of \underline{L}' is very nearly exact.

Consider now, a second coil, having \underline{N} turns, arranged in the same general way as were those in the first coil. Suppose this coil to have a smaller diameter than the first coil, and, to

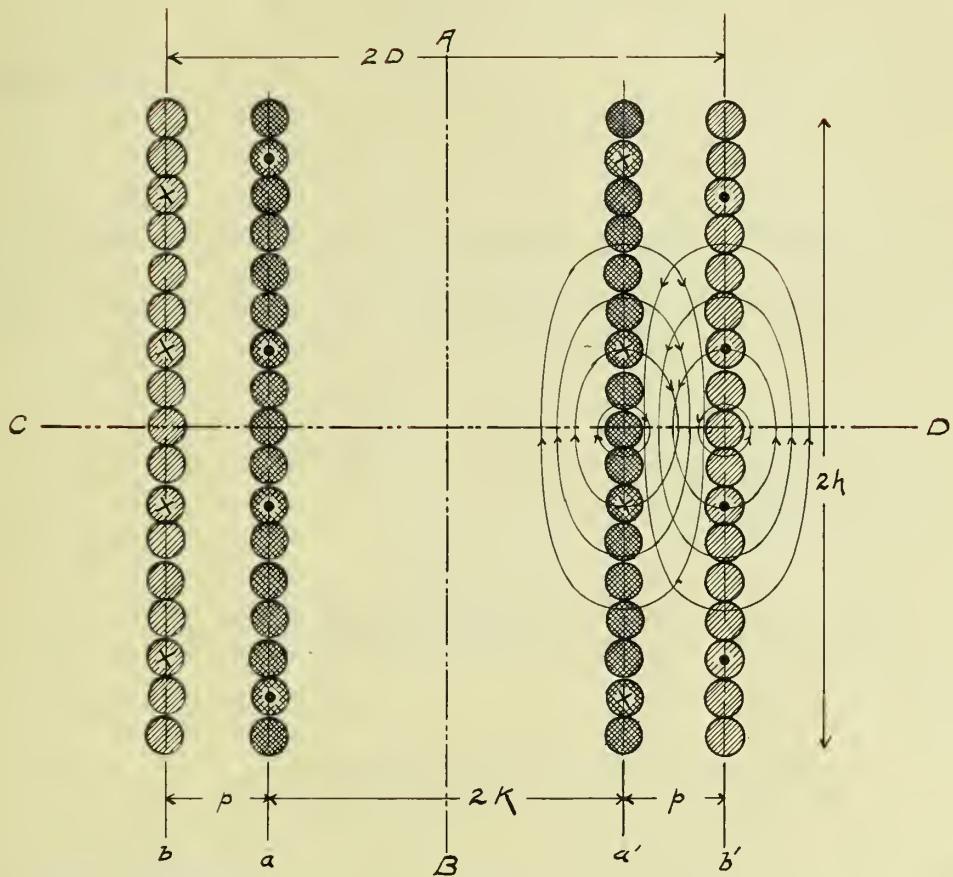


FIGURE 31.

be placed concentrically within the first coil. Let a current I flow thru both coils; but suppose that the direction of this current in the second coil is opposite to that in the first coil. This arrangement is shown in Figure 31.

Returning to equation (35)

$$S_v = 0.8 \pi n I y \arctan \frac{h}{D}$$

it is evident that

$$S_\alpha = \int_0^h 2\pi (D-y) S_v dy$$

is the total flux across a right section at \underline{CD} , due to the primary, (the coil bb'), which does not interlink with the secondary, (the coil, aa'). Then

$$S_\alpha = \int_0^h 0.8 \pi n I \arctan \frac{h}{D} \times y (D-y) 2\pi dy \quad (44)$$

and

$$\begin{aligned} S_\alpha &= \frac{1.6 \pi^2 N I}{2h} \arctan \frac{h}{D} \left[-\frac{y^2 D}{2} + \frac{y^3}{3} \right]_0^h \\ &= \frac{0.8 \pi^2 N I}{h} \left[\frac{h^3}{3} - \frac{Dh^2}{2} \right] \arctan \frac{h}{D} \end{aligned} \quad (45)$$

Now, in the same manner as that in which formula (31) was derived, the value of the stress intensity at a point on \underline{CD} , Figure 27, outside the coil, may be found, and the resulting equation is

$$\begin{aligned} S'_\omega &= \frac{0.4 I N y}{\pi} \left[\arctan \frac{h}{y} - \arctan \frac{h}{2D+y} \right] \\ S'_\omega &= \frac{0.4 I N y}{\pi} \left[\arctan \frac{2Dh}{y(2D+y)+h^2} \right] \end{aligned} \quad (46)$$

and by the application of the method used in the case of equation

$$(35) \quad S'_\omega = \frac{0.4 I N}{2\pi h} y \arctan \frac{2Dh}{3D^2+h^2} \quad (47)$$

Now, if D in equation (47) is the radius of the smaller coil and is equal to K, then the total flux outside the smaller coil, passing thru the right section taken on the line CD, and not interlinked with the larger coil, is

$$S_p = \int_0^h 2\pi(K+y) 5\omega dy$$

and

$$S_p = \frac{0.417\pi}{4} \left[\frac{K\beta^2}{2} + \frac{\beta^3}{3} \right] \operatorname{arc} \tan \frac{2Kh}{3K^2 + h^2} \quad (48)$$

Now the ratio, $\frac{S_K}{S_p}$, multiplied by L', should give the total leakage inductance of the primary or outside coil, since $\frac{S_K}{S_p}$ should be the part of the total primary leakage inductance due to the lines of unit stress set up in the space between the coils. Then

$$L''' = \frac{S_K}{S_p} \left[\frac{5.23 N^2 D^3}{4} \left[\operatorname{arc} \tan \frac{h}{D} - \operatorname{arc} \tan \frac{2h}{D} - \frac{D}{4h} \log \left(1 + \left(\frac{2h}{D} \right)^2 \right) \right] \right]$$

$$L''' = \frac{5.23 N^2}{4} \frac{[2\beta^3 - 3D\beta^2]}{\left[\operatorname{arc} \tan \frac{h}{D} - \frac{1}{2} \operatorname{arc} \tan \frac{2h}{D} \right]} \left[\operatorname{arc} \tan \frac{h}{D} - \operatorname{arc} \tan \frac{2h}{D} - \frac{D}{4h} \log \left(1 + \left(\frac{2h}{D} \right)^2 \right) \right] \quad (49)$$

and if $\operatorname{tan} \theta = \frac{h}{D}$, and $\operatorname{tan} \phi = \frac{2h}{D}$,

$$L''' = \frac{10.46 N^2}{4} \frac{[2\beta^3 - 3D\beta^2]}{2\theta - \phi} \left[\theta - \phi - \frac{1}{2} \cot \phi \log \sec^2 \phi \right] \quad (50)$$

It is evident, that by the substitution of K for D in equations (39) and (41), they can be made to apply equally well for the smaller coil, that is, the secondary coil. Then

$$S_{os} = \frac{1.305 NIK^3}{4} \left[\operatorname{arc} \tan \frac{h}{K} - \frac{1}{2} \operatorname{arc} \tan \frac{2h}{K} \right]$$

and

$$L'_s = \frac{5.23 N^2 K^3}{4} \left[\operatorname{arc} \tan \frac{h}{K} - \operatorname{arc} \tan \frac{2h}{K} - \frac{K}{4h} \log \left(1 + \left(\frac{2h}{K} \right)^2 \right) \right]$$

Then in the same manner as that in which equation (49) was developed,

an expression for the total leakage inductance, L'' , due to the secondary coil may be obtained. That is, $\frac{S\beta}{S_{DS}} L'_s$ should be this value, or

$$L'' = \frac{S\beta}{S_{DS}} L'_s$$

Then substituting, and placing

$$\begin{aligned} \tan \gamma &= \frac{h}{K} & \tan \delta &= \frac{2h}{K} & \tan \gamma &= \frac{2Kh}{3K^2 + h^2} \\ L'' &= \frac{532N^2}{h} \frac{[3Kp^2 + 2p^3]\gamma}{2y - \delta} \left[y - \delta - \frac{K}{4h} \log \sec^2 \delta \right] \end{aligned} \quad (51)$$

Then the total leakage inductance due to the two coils is

$$L_T = L''' + L''$$

and

$$L_T = \frac{N^2}{h} \left[\frac{10.46[30p^2 - 2p^3]\theta}{2\theta - \varphi} \left[\theta - \varphi - \frac{D}{4h} \log \sec^2 \varphi \right] + \frac{532[3Kp^2 + 2p^3]\gamma}{2y - \delta} \left[y - \delta - \frac{K}{4h} \log \sec^2 \delta \right] \right] \quad (52)$$

If the flux thru the coils, or the stresses set up therein are due to an alternating current having a frequency of f cycles per second, then the leakage reactance of the two coils is

$$X = 2\pi f L_T$$

or, for convenience

$$X = \omega L_T$$

Such is the development of the expression for the leakage inductance for concentric cylindrical coils having air cores. The application of the theory of this development to the actual transformer will now be made.

In the discussion to this point, consideration has been given only to coils having air cores, and having no traces of iron about them. Coils having an iron core which forms part of a closed magnetic circuit will now be considered. Suppose the two coils in Figure 32, to have each the same number of turns, and that the current I , flows thru the coils in opposite directions. The iron circuit has a permeability very much greater than that of air; that is, the magnetic stresses are set up with very much greater facility in iron, than they are in the air. Consequently, the general form of the field of stress under such conditions is very different from that existing when no iron is present. This fact complicates the determination of the coefficient of leakage inductance, L_T , very markedly.

It has been shown experimentally, that the variation of the total leakage flux thru the coil with the distance from the center, is rectilinear, and may be represented by lines ef , and gh , in Figure 32. If then, as in the theory for air core coils, the leakage lines of unit stress passing across the right section at CD , may be determined, the actual leakage inductance may be obtained by the interlinkage of this value of stress across the section with the proper number of turns. In any arrangement of coils, such as is shown in Figure 32, having an iron path so near to the end turns, as is usual with core type transformers with cylindrical coils, the determination of the magnetic stress intensity set up in any section such as CD , is dependent upon a large number of variables. Some of these are: the distance of the inner surface of the coil from the iron, the distance of the ends of the coil from the iron, the char-

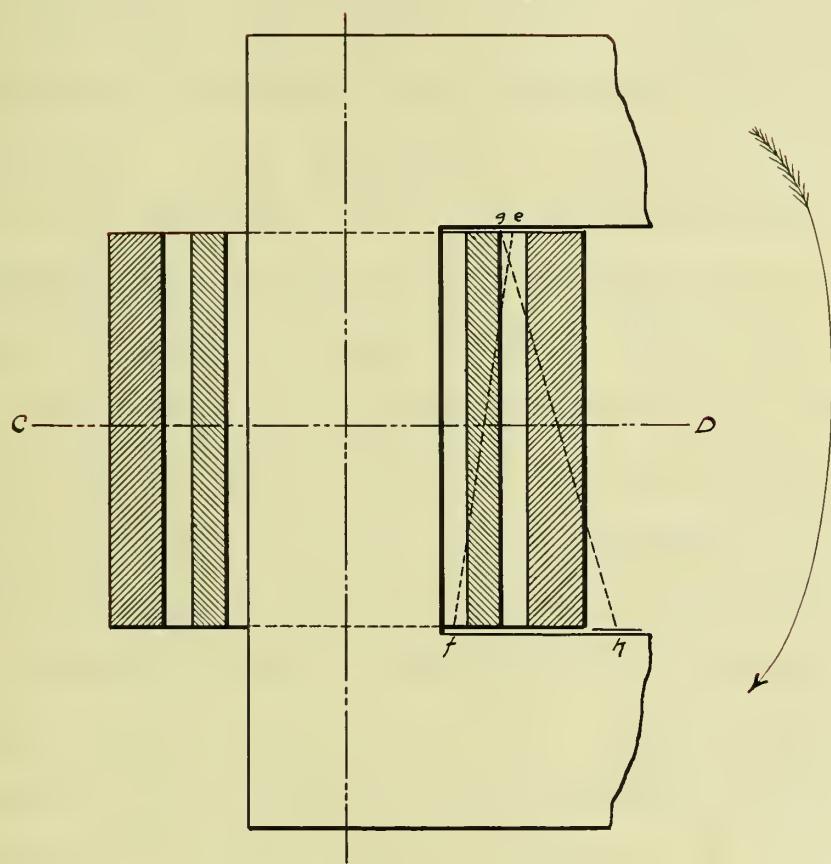


FIGURE 32.

acter of the medium outside the coil, and so forth. This necessitates obtaining a very complex mathematical result of the determination of the inductance coefficient, if any great degree of accuracy is desired. Consequently no attempt will be made to obtain formulae for coils having iron cores by the general method employed in the case of coils having air cores. A different method of attack, however will be followed, and formulae will be obtained, by means of which, the leakage inductance can be obtained with a fairly high degree of accuracy. Three arrangements of the coils and the iron circuit will be considered: The core type transformer, and the shell type transformer, both having cylindrical coils, and a type of transformer having flat coils, placed one above the other.

For the first of these considerations, Figure 33 will be employed. Suppose two cylindrical coils \mathcal{A} , and \mathcal{B} , to be placed on one part of a closed magnetic circuit as shown. Also, suppose a corresponding arrangement on the opposite side of the line \mathcal{CD} . This is the arrangement of the simple core type transformer. Suppose coil \mathcal{A} , has turns N , and that coil \mathcal{B} has turns N_2 ; also, that I_1 amperes flow thru \mathcal{A} , and that I_2 amperes flow thru \mathcal{B} in the opposite direction from that of the current in \mathcal{A} . The length of each coil is h , and the coils \mathcal{A} and \mathcal{B} have the thicknesses s and P , respectively.

From an investigation of Figure 33, it is evident that the path of the lines of stress which pass thru coil \mathcal{A} , is composed of iron and air or other nonmagnetic substance, and that the part of the path which lies in the iron has negligible reluctance since it offers such great facility for the setting up of the magnetic stresses. Then the length of path having appreciable reluctance, is that of the air path or is h . In the same way, the length of the path

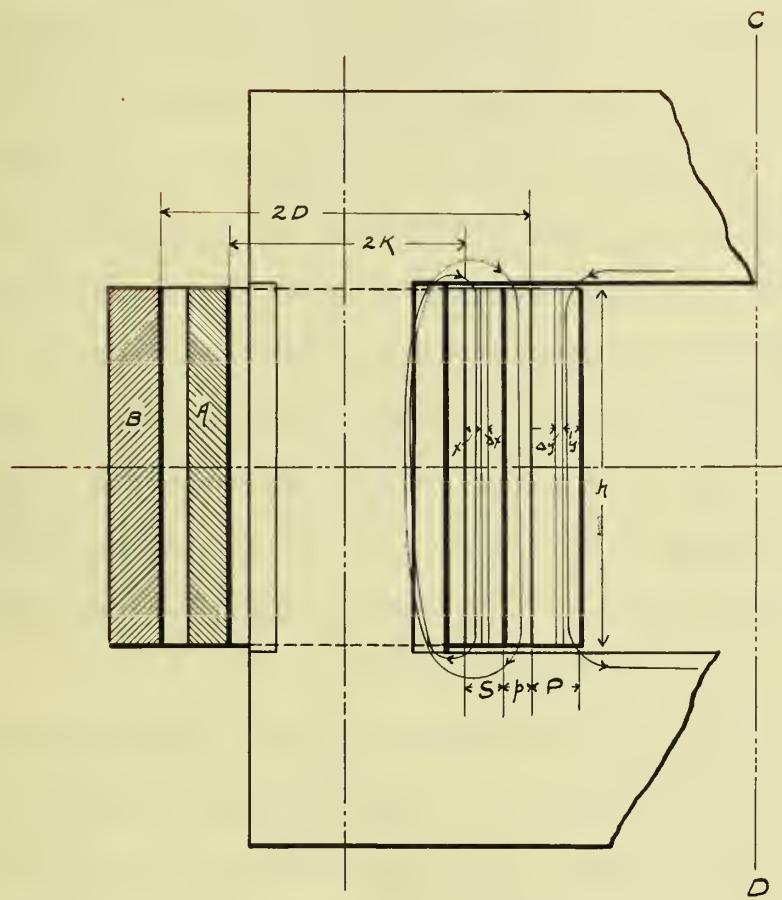


FIGURE 33.

of the lines thru the coil 3, and thru the space between the coils, may be shown to be h. This assumption holds directly for the space under the iron yoke, that is, the space om, Figure 34. In the space obm, however the assumption does not hold, and in this development, the average length will be taken as kh, where k' is greater than unity, and has slightly different values for transformers of this type but which differ in general arrangement of coils. Since the effect of l lines of stress encircling $\frac{1}{h}$ of the coil is the same as that of $\frac{p}{h}$ lines encircling all the turns of the coil, when the determination of inductance is considered such an assumption will be made, in order that the differential leakage flux may be integrated across the coil. If such assumptions are made, it follows that the flux will have the general path shown in Figure 33, by the lines with the arrow-heads. Now consider the turns of coil 2, which are in the layers in the section x of the coil. The number of turns is $\frac{xN}{s}$, and the total magnetomotive force due to these turns is

$$M_{mf} = 0.4\pi \frac{x}{s} N I$$

Now this magnetomotive force impressed upon the path of width dx and length, kh, which extends completely around the part of the coil $\frac{x}{s}$, so that its cross section is $2\pi(k+x)dx$, where the radius of the coil is k, produces in this path a differential of flux,

$$\begin{aligned} d\phi &= \frac{0.4\pi \frac{x}{s} N I \times 2\pi(k+x)dx}{kh} \\ &= \frac{0.8\pi^2 N I}{kh s} (k+x) \times dx \end{aligned} \quad (53)$$

This differential of flux or stress is interlinked with $\frac{xN}{s}$ turns of coil 2. Then the differential inductance of this part of the coil due to itself is

$$dL = \frac{0.8\pi^2 N^2}{s^2 h k'} (k+x) \times^2 dx \quad (54)$$

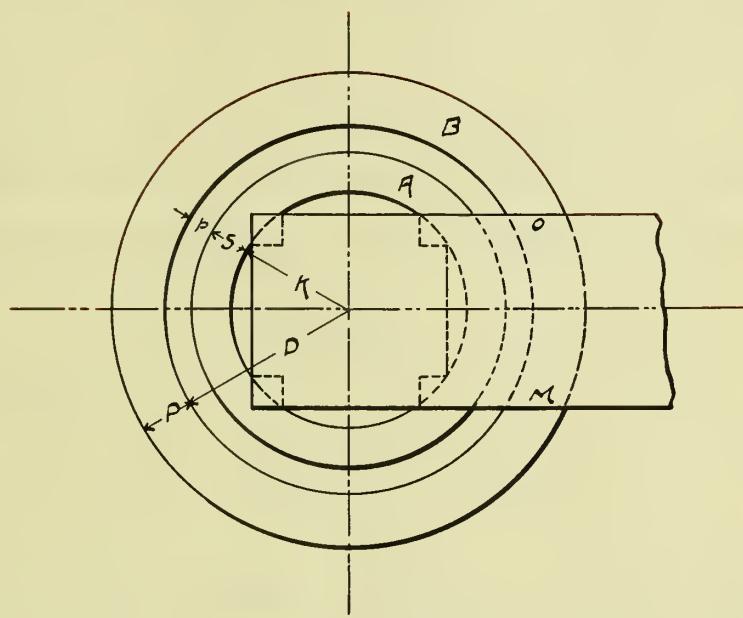


FIGURE 34.

and if this differential value is integrated from zero to s, or across the coil,

$$\begin{aligned} L_q &= \int_0^s \frac{0.8\pi^2 N_1^2}{s^2 h k'} (\kappa + x) x^2 dx \\ &= \frac{0.8\pi^2 N_1^2}{k' h} \left[\frac{\kappa s}{3} + \frac{s^3}{4} \right] \end{aligned} \quad (55)$$

and L_q is the total leakage inductance of the coil q, due to itself.

In the same way, a value for L_B may be obtained. That is, if $\frac{y N_2}{P}$ turns produce a differential flux in a path of length k' h, and cross section $\frac{2\pi(0+y)dy}{2}$, the inductance increment is

$$dL_B = \frac{0.8\pi^2 N_2^2}{k' P^2 h} (0+y) y^2 dy$$

and between zero and P,

$$L_B = \frac{0.8\pi^2 N_2^2}{k' h} \left[\frac{0 P}{3} + \frac{P^3}{4} \right] \quad (56)$$

Now, there is impressed upon the path formed by the space between the coils, a magnetomotive force equal to the total magnetomotive force of either one of the two coils, that is, either 0.4πN₁I₁, or 0.4πN₂I₂. Also, the length of this path is k' h, and its cross section is $\frac{2\pi(\kappa+s+0)P}{2}$. Then the flux thru this space is

$$\begin{aligned} \phi_L &= \frac{0.4\pi N_1 I_1 \times 2\pi(\kappa+s+0)P}{2 k' h} \\ &= \frac{0.4\pi^2 N_1 I_1 (\kappa+s+0)P}{k' h} \end{aligned}$$

which is based upon the assumption that the ampere turns on both coils are the same. This condition is very closely obtained in the modern transformer. Now, $\frac{\phi_L}{2}$, interlinks with N₂, and also with N₁, turns, then all of φ_L may be considered to interlink with either N₂, or N₁, turns. Then the inductance due to the lines of stress

which pass thru this space, is

$$L_L = \frac{0.7 \pi^2 N_1^2 (K+5+D) \rho}{K' h} \quad (57)$$

Then the total leakage inductance of the arrangement, considering also, the two coils on the part of the iron circuit not shown in the figure, is

$$L = 2 \frac{0.8 \pi^2}{K' h} \left[N_1^2 \left(\frac{Ks}{3} + \frac{s^2}{4} \right) + N_2^2 \left[\frac{DP}{3} + \frac{P^2}{4} \right] + \frac{N_1^2}{2} [D + K + 5] \rho \right] \quad (58)$$

If N_1 does not equal N_2 , the effective inductance of coil B , may be found by multiplying L_B by $\left(\frac{N_1}{N_2} \right)^2$, and the result may be inserted in equation (58), so that in general,

$$L_C = \frac{1.6 \pi^2 N_1^2}{K' h} \left[\frac{Ks}{3} + \frac{s^2}{4} + \frac{DP}{3} + \frac{P^2}{4} + \frac{D^2 - (K+5)^2}{2} \right] \quad (59)$$

This method of development will now be applied to a transformer of the shell type. Consider Figure 35. Suppose a current I_1 , to flow in the same direction, in coils A , and C , each of which has N_1 turns. Also, suppose a current I_2 , to flow in the opposite direction in coil B , which has N_2 turns. Now, in the same manner as that employed in the preceding discussion, except that a higher degree of accuracy is a characteristic, it can be shown that the parts of the leakage paths which have appreciable reluctance, are all equal, in length, to $K' h$, where K' is very nearly unity. Then, the leakage inductance due to coil C is, as before,

$$L_C' = \int_0^P \frac{0.8 \pi^2 N_1^2}{s^2 h K'} (D+x) x^2 dx$$

and

$$L_C = \frac{0.8 \pi^2 N_1^2}{K' h} \left[\frac{DP}{3} + \frac{P^2}{4} \right] \quad (60)$$

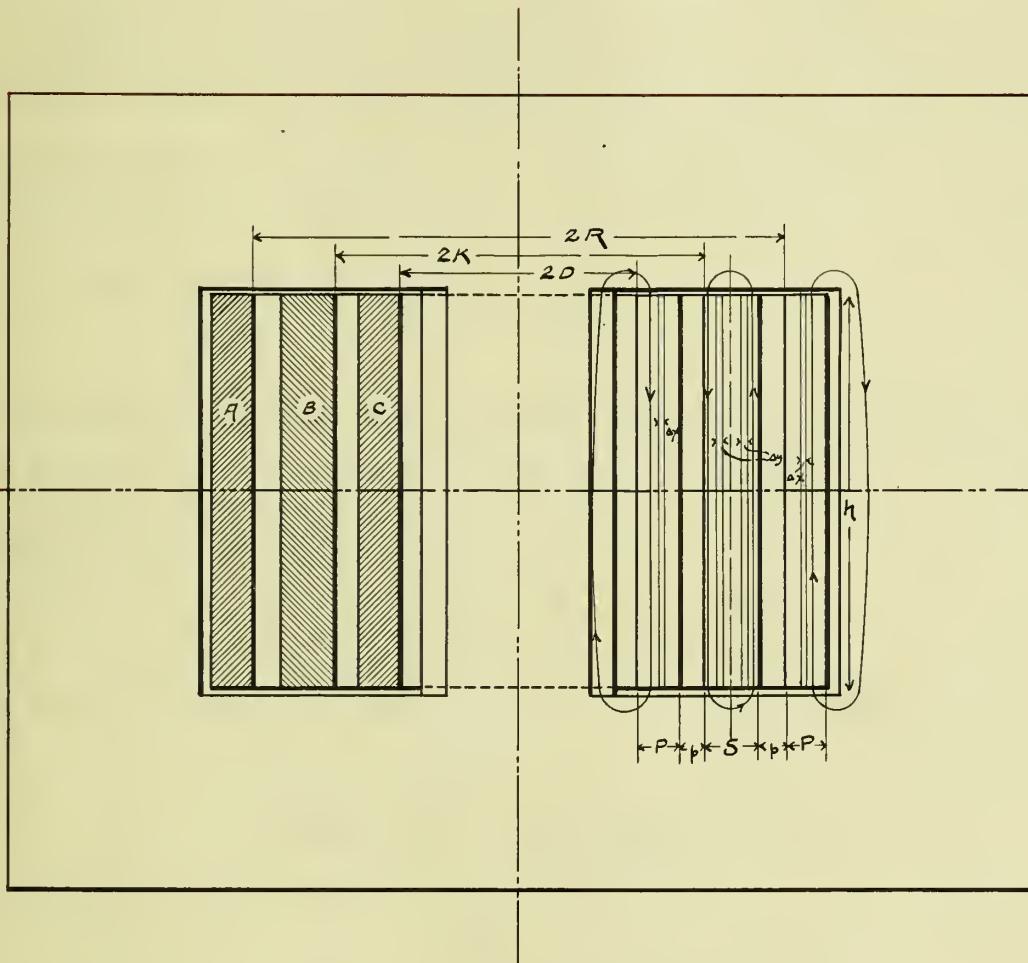


FIGURE 35

By the same method, the value of L_A , may be found to be

$$L_A = \frac{0.8 \pi^2 N_1^2}{K'h} \left[\frac{R^3}{3} + \frac{P^2}{4} \right] \quad (61)$$

Now consider the coil B. The leakage inductance of this coil due to itself, may be found by integration from $\frac{-s}{2}$ to $\frac{s}{2}$, of an expression similar to those preliminary to equations (60) and (61). Consider a section of width $2x$ at the center of the coil. suppose that outside of this section, there are two other sections of width dx each, the length of all sections being h . The turns in the center section are $\frac{2x}{5} N_2$, and these turns produce a magnetomotive force of value

$$0.8 \pi N_2 I_2 \times \frac{x}{5}$$

This magnetomotive force may be considered to be impressed upon a path of length $2Kh$, and area of cross section $\pi(2K+s)dx$. Then the total number of lines of stress set up in this path, is

$$\phi = \frac{0.8 \pi N_2 I_2 x}{s} \times \frac{2\pi(2K+s)}{Kh} dx$$

and the interlinkages of this flux with the turns $\frac{2x}{5} N_2$, per unit current are

$$dL_B = \frac{3.2 \pi^2 N_2^2 (2K+s) x^2 dx}{K'h s^2}$$

and

$$L_B = \int_0^{\frac{s}{2}} \frac{3.2 \pi^2 N_2^2 (2K+s) x^2 dx}{K'h s^2}$$

$$= \frac{0.4 \pi^2 N_2^2}{K'h} \left(\frac{2Ks}{3} + \frac{s^2}{3} \right) \quad (62)$$

Now consider the spaces between the coils. Due to coil B, there is impressed upon the two spaces, a magnetomotive force of $0.4\pi N_2 I_2$, considering the spaces in series, to form an air path of length $2hk'$

This has the same effect as though, due to coil c or coil g, a magnetomotive force of 0.4πNI, is impressed upon each space separately. Now, the cross section of this path is 2π(2K+5)h, and the flux thru it is

$$\phi = \frac{0.4 \pi N_2 I_2}{2hK} \times 2\pi(2K+5)h$$

and all of this flux is considered to interlink with the turns N₂.

Then

$$L_2 = \frac{0.4 \pi^2 N_2^2}{K'h} [2K+5]h \quad (63)$$

Then the total leakage inductance of the transformer is, when reduced entirely to "N values" or to the primary

$$L^s = \frac{0.2 \pi^2 N_3^2}{K'h} \left[\frac{DP}{3} + \frac{P^2}{4} + \frac{R_P}{3} + \frac{P^2}{4} + \frac{4Ks}{3} + \frac{2s^2}{3} + 2(2K+5)h \right] \quad (64)$$

where N₃=2N, and is the total number of turns in which the current I₁, flows. Also

$$L^s = \frac{0.2 \pi^2 N_3^2}{K'h} \left[\frac{P}{3} (R+P) + \frac{P^2}{2} + \left(\frac{2s^2}{3} + 2h \right) (2K+5) \right] \quad (65)$$

Next will be considered an arrangement employing round and thin coils, called technically "flat coils", placed upon an iron circuit, as shown in Figure 36. Suppose coils g, c, and so forth, to have each N₁/n turns, and that a current I₁ passes thru all of them in the same direction. Also suppose coils B, D, and so forth to have N₂/n-1 turns each, and that a current I₂, passes thru all these turns in a direction opposite to that of I₁. For convenience, it will be considered that the general dimensions of the arrangement are represented by the symbols in Figure 36. Due to the position of the coil g, and that of the one in a corresponding position on the op-

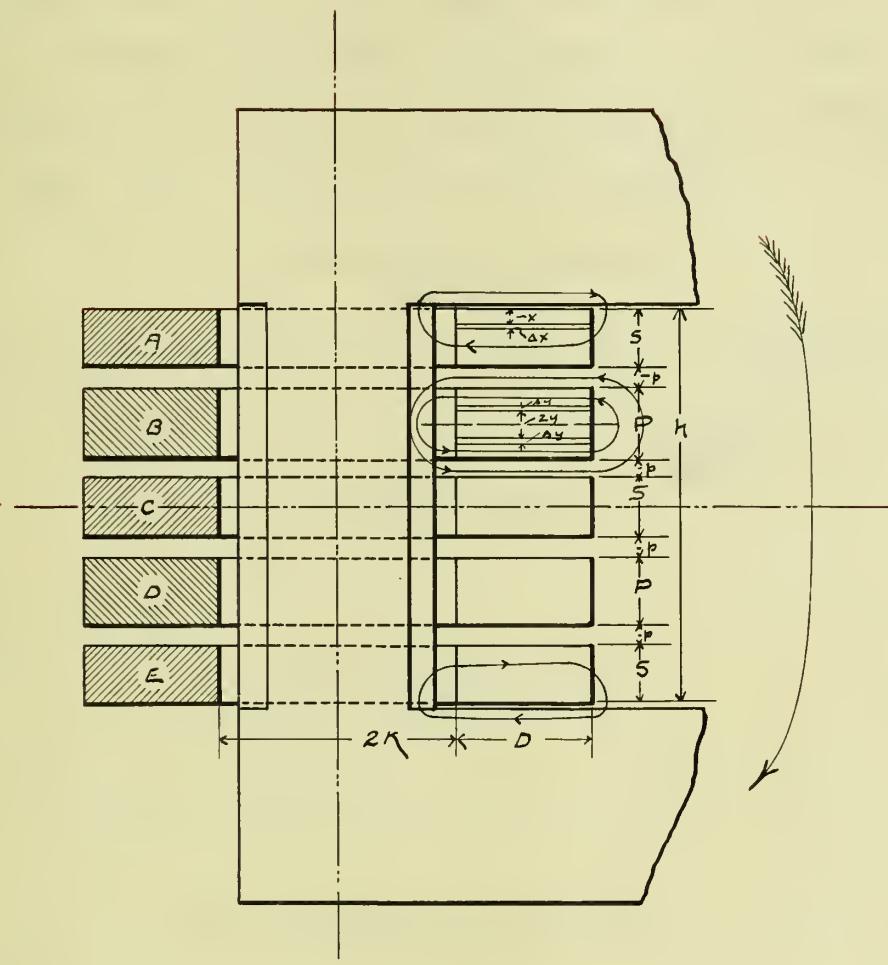


FIGURE 36.

posite end of the leg, called the "end coils", the determination of the inductance of these coils must be made in a manner different from that to be employed for the other coils. Consider coil 7. The flux thru this coil may be assumed to have the path shown by the arrow line. Now consider the small part of the coil, of thickness x. This part contains a number of turns $\frac{xN_1}{57}$, and these set up a magnetomotive force of $\frac{0.8\pi N_1 I}{57}$ units. Now, this magnetomotive force is impressed upon a path having, as an average cross section $\frac{2\pi(2K+D)dx}{2}$, and as a length, K'D. Then the flux thru this path is

$$\phi = \frac{0.8\pi^2 N_1 I (2K+D) x dx}{K'' 57 D}$$

and this flux is interlinked with $\frac{xN_1}{57}$ turns. The total leakage inductance of this coil, due to itself, is then

$$\begin{aligned} L_7 &= \int_0^S \frac{0.8\pi^2 N_1^2 (2K+D)}{K'' 57^2 H^2 D} x^2 dx \\ &= \frac{0.8\pi^2 N_1^2 (2K+D) S}{3K'' H^2 D} \end{aligned} \quad (66)$$

And the total leakage inductance due to the four coils in this condition is

$$L'_7 = \frac{3.2\pi^2 N_1^2 (2K+D) S}{3K'' H^2 D} \quad (67)$$

Now consider coil 6. Choose a portion in the center of the section shown, parallel to the long dimension of the section. Suppose this portion to have a width 2x, and a length D. It includes $\frac{2xN_1}{57}$ turns and has $\frac{0.8\pi N_1 I x}{57}$ units of magnetomotive force. This magnetomotive force may be considered to be impressed upon a path which has an average cross section of $\frac{\pi(2K+D)dx}{2}$ units and a length of 2Kb units. Then the total flux thru this path is

$$\phi = \frac{0.4 \pi^2 N_1 I_1 (2K+D) \times dx}{K'' S H D}$$

and this flux is interlinked with $\frac{2 \times N_1}{S_1}$ turns. The total leakage inductance of coil C, due to itself, is then,

$$\begin{aligned} L_C' &= \int_0^S \frac{0.8 \pi^2 N_1^2 (2K+D) x^2 dx}{K'' S^2 H^2 D} \\ &= \frac{0.8 \pi^2 N_1^2 (2K+D) S}{3K'' H^2 D} \end{aligned} \quad (68)$$

and due to the $2(n-2)$ coils of this nature, on the transformer, the total leakage inductance is

$$L_C' = \frac{1.6 \pi^2 N_1^2 (2K+D) S}{3K'' D} \frac{n-2}{n^2} \quad (69)$$

By the same method, the leakage inductance of the coils B, D, and so forth, is found to be

$$L_B' = \frac{1.6 \pi^2 N_1^2 (2K+D) \rho}{3K'' (n-1) D} \quad (70)$$

Now, consider the spaces between the coils A and B, and between the coils B and C. It may be assumed that the total magnetomotive force of B is impressed upon a path, which consists of these two spaces in series, and which has a length $2K\rho$ and an average cross sectional area $\frac{2\pi(2K+D)\rho}{2}$. Now, the total magnetomotive force of coil B, is $\frac{0.4\pi N_2 I_2}{n-1}$, and the total flux thru the path under consideration, is

$$\phi = \frac{0.4 \pi^2 N_2 I_2 (2K+D) \rho}{2D(n-1)K''}$$

This flux interlinks with $\frac{N_2}{n-1}$ turns of B, and consequently, the total leakage inductance of this flux for the $2(n-1)$ such paths of the transformer, is

$$L_B' = \frac{0.8 \pi^2 N_2^2 (2K+D) \rho}{2D(n-1)K''} \quad (71)$$

Then the total leakage inductance of the transformer is

$$L^T = \frac{0.4\pi^2 N^2 (2K+D)}{D} \left[\frac{45(2+nK''-2K'')(n-1) + K''n^2(4P+3b)}{3K''n^2(n-1)} \right] \quad (72)$$

which is obtained from the formula

$$L^T = L'_A + L'_B + L'_C + L'_L$$

and all the inductance formulae give results in henries, provided dimensions are measured by the C.G.S. system.

No attempt will be made to apply this method to the shell type transformer with flat coils, since the formula (72) holds for this condition also, within a reasonable degree of accuracy.

It is to be noted that equations (59), (65), and (72), are expressions for L, the leakage inductance of the transformer under consideration. From the method development, it is evident that the lines of stress to which these values of L are due, are the lines which form the stray or leakage fields about the windings, which were mentioned earlier in the paper; and that L is the actual coefficient of leakage inductance which, when multiplied by the frequency factor, $2\pi f$, the current, and 10^{-9} , gives the potential drop thru the transformer in volts, when the resistance drop is considered to be negligible. These equations will be discussed later in the paper.

Consideration will now be given to the relations which this leakage inductance factor L, bears to the other characteristic factors of the transformer. For convenience

$$L = L_p + L_s$$

where L_p is the leakage factor of the primary winding, and L_s , that of the secondary winding. These may be expressed as

$$S = S_p + S_s$$

in order that they may be distinguished from the factors of mutual inductance, which may be used, later. Let E_1 and E_2 denote the electromotive force impressed upon the primary winding, and that appearing at the terminals of the secondary winding, respectively. Suppose R_p to be the resistance of the primary winding, R_s that of the secondary winding, and R_o that of the secondary circuit outside the transformer. Let M be the mutual inductance of the primary upon the secondary, or vice versa, assuming that it is the same from either point of view, when reduced by the proper factor. Also, consider that the primary has N_1 turns, with I_1 amperes flowing thru them; and that the secondary has N_2 turns with I_2 amperes flowing therein. Now considering the four fields of stress given on Page 25, it is evident that S_p corresponds to the field #1, the primary singly interlinked field; that S_s corresponds to the secondary interlinked field, #3; and that M corresponds to either field #2, or field #4, the primary, or the secondary doubly interlinked field. Then if the secondary circuit is open

$$\begin{aligned} E_1 &= \left[\frac{N_1}{N_2} M + S_p \right] \frac{dI_1}{dt} \\ E_2 &= M \frac{dI_1}{dt} \quad I_2 = 0 \end{aligned} \tag{73}$$

Reducing E_2 to primary consideration, and subtracting

$$E_1 - E_2 \frac{N_1}{N_2} = 2\pi f I_1 S_p \tag{74}$$

where the electromotive forces are now effective values. (Sine waves are assumed and losses neglected for convenience, at present). And

$$E_2 - E_1 \frac{N_2}{N_1} = 2\pi f I_2 S_p \tag{75}$$

Now, if the secondary coil is short circuited,

$$\dot{E}_1 = \frac{N_1}{N_2} M \frac{dI_1}{dt} + S_p \frac{dI_1}{dt} + M \frac{dI_2}{dt} \quad (76)$$

$$0 = M \frac{dI_1}{dt} + \frac{N_2}{N_1} M \frac{dI_2}{dt} + S_s \frac{dI_2}{dt}$$

Now, from the first equation of (76)

$$\dot{E}_1 = \frac{N_1}{N_2} M \frac{dI_1}{dt} + S_p \frac{dI_1}{dt} - M \frac{1}{\frac{N_2}{N_1} + \frac{S_s}{M}} \frac{dI_2}{dt} \quad (77)$$

and reducing again to effective values (using sine waves)

$$E_1 = 2\pi f I_1 \left[S_p + \frac{N_1^2}{N_2^2} S_s \left(1 + \frac{S_s}{L_s} \right) \right]$$

where L_s is the inductance factor of the secondary doubly inter-linked stresses referred to the secondary. Then

$$E_1 = 2\pi f \left[S_p I_1 + \frac{N_1}{N_2} S_s I_2 \right] \quad (78)$$

and the impressed voltage at short circuit is equal to the sum of the leakage voltages. This equation becomes of value in the actual tests.

Consideration will now be given to transformers having leakage, and iron, and copper, losses. If the secondary is short circuited and normal secondary load current flows, it is evident that the iron losses are very small, due to the very low density in the iron, and consequently, the influence of these losses upon the leakage potential may be neglected. Then, at short circuit, letting $\omega = 2\pi f$, for convenience

$$E_1 \sin(\omega t + \psi) = \left(\frac{N_1}{N_2} M + S_p \right) \frac{dI_1}{dt} \sin \omega t + M \frac{dI_2}{dt} \sin(\omega t + \alpha) + K_1 I_1 \sin \omega t$$

and

(79)

$$0 = \left(\frac{N_2}{N_1} M + S_s \right) \frac{dI_2 \sin(\omega t + \alpha)}{dt} + \frac{M dI_1 \sin \omega t}{dt} + r_2 I_2 \sin(\omega t + \alpha)$$

Then, from the secondary equation in (79),

$$\tan \alpha = \frac{r_2}{\omega \left(\frac{N_2}{N_1} M + S_s \right)} \quad (80)$$

and

$$\frac{I_1}{I_2} = \left[\left(\frac{N_1}{N_2} + \frac{S_s}{M} \right)^2 + \left(\frac{R_2}{\omega M} \right)^2 \right]^{\frac{1}{2}} \quad (81)$$

Then from (79),

$$\begin{aligned} E_1 \sin(\omega t + \psi) &= \omega I_1 \left[\frac{N_1}{N_2} M + S_p - \frac{\frac{N_2}{N_1} M + S_s}{\left(\frac{N_2}{N_1} + \frac{S_s}{M} \right)^2 + \left(\frac{R_2}{\omega M} \right)^2} \right] \cos \omega t \\ &+ I_1 \left[r_1 + \frac{r_2}{\left(\frac{N_2}{N_1} + \frac{S_s}{M} \right)^2 + \left(\frac{R_2}{\omega M} \right)^2} \right] \sin \omega t. \end{aligned} \quad (82)$$

If it is desired to obtain the sum of the leakage potentials or of the electromotive forces, due to L or S, the leakage inductance factor, it is necessary to assume,

(1) That r_2^2 may be neglected when compared with $(\omega L_2)^2$, which holds for modern transformers;

(2) That S_s may be neglected when compared with L_2 , which holds for modern transformers.

Then, if $N_1 I_1 = N_2 I_2$, which is very nearly true, and, may be considered to be true for all practical calculations, at short circuit $\alpha = \pm \pi$

$$E_2 \sin \omega t = - \frac{N_2}{N_1} M \frac{dI_2 \sin \omega t}{dt} + \left(\frac{N_2}{N_1} M + S_s \right) \frac{dI_1 \sin \omega t}{dt} + r_2 I_2 \sin \omega t.$$

and also

$$E_2 = I_2 \left[(\omega S_s)^2 + r_2^2 \right]^{\frac{1}{2}} \quad (83)$$

and in the same manner,

$$E_s = I_s \left[(\omega s_p)^2 + r^2 \right]^{\frac{1}{2}} \quad (84)$$

Therefore, the leakage fields, alone, are present when $\frac{N_1}{N_2} = \frac{I_2}{I_1}$; and also there exists under this condition, a phase difference between the two currents, of 180 degrees.

A question arises at this point, as to whether or not, the leakage at normal load on the transformer and consequently, at normal magnetization of the iron, is the same as that at short circuit, when the magnetization is at very low density. This brings up the comparison of the leakage paths when compared with the total flux path thru the transformer; since the latter is almost entirely in the iron, while the former is, to a fairly great extent, in the air. Consequently, the leakage and its corresponding potential drop, depends upon the permeability μ , of the iron. Under the conditions of normal operation, the permeability of the iron is between 2000 and 5000. However, at short circuit, the intensity is very low as is the permeability also, since the iron in ordinary transformers is then operated around, or at, the lower knee of the curve. That is, between points o and q, or at point q, in Figure 37. Then, if for the leakage, the air path comes seriously into consideration, the self induction at short circuit need not be exactly the same as that at normal operation. An accurate determination of the leakage inductance, cannot be made, therefore, by means of the short circuit test; but, for practical purposes, the error is sufficiently small to be neglected, and, if care is used, the short circuit test may be utilized.

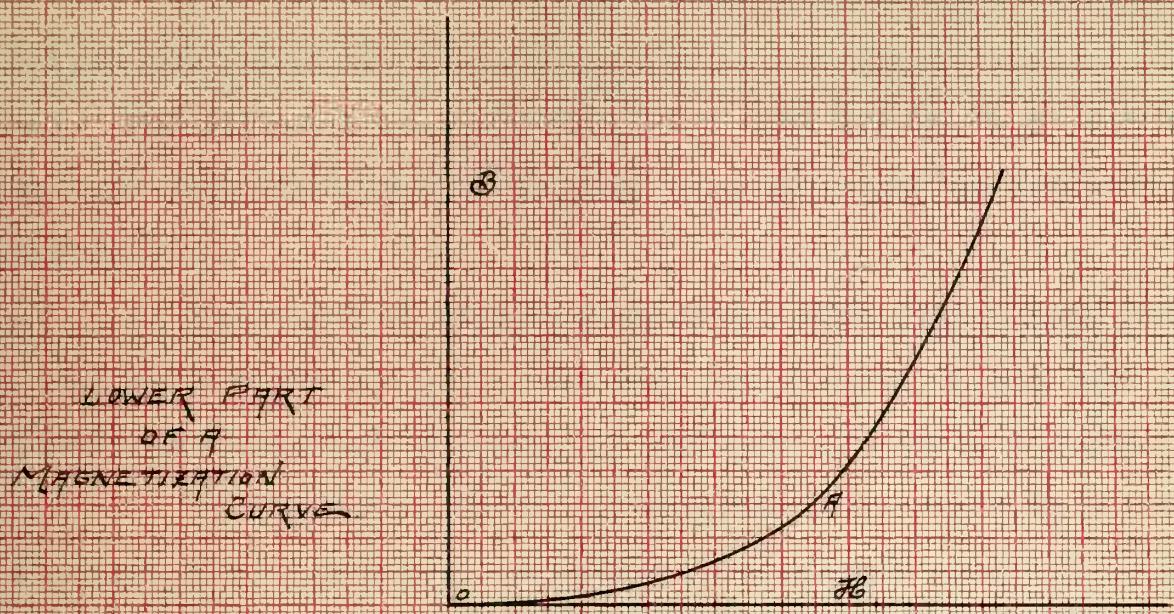


FIGURE 37.

From the foregoing theory, certain conclusions may be drawn. In discussing these, most consideration will be given the conclusions obtained from the latter part of the development. The general theory of the earlier portion of the paper, while of importance for the deduction of formulae which are later employed, and for justifying the assumptions made, is not of practical interest in the discussion of transformers.

From this theory however, there are certain important deductions to be made. It is clear, from the general development, that this development depends upon the theory of a field of stress in the medium surrounding any conductor; and that, unless this theory (based upon tubes of unit stress), holds, a lucid explanation of the behavior of the lines of stress cannot be given. Of the general conclusions from this part of the theory, the following are of import-

ance:

The lines of stress represent tubes of unit stress which completely pervade the medium around a conductor carrying a current. The stresses are distributed in this medium in a definite manner, and this distribution follows laws which may be expressed mathematically.

The lines of stress never cross each other; they are always set up in a closed path; and they are always, during both the stationary and transient periods, in the form of closed figures.

If the fields of stress are due to alternating current, the stresses are alternately increased to a maximum positive value and reduced thru zero to a negative maximum value, depending upon the alternations of the exciting current.

Due to the attempt of Nature to preserve an equilibrium, there are induced in wires carrying a current, a "back", or counter electromotive force, during the periods of change of the current in the wire and of the fields of stress around the conductor.

If two wires are placed near each other, the change of the field of stress due to one of the wires, produces a change in the characteristics of the other thru the induction of an electromotive force in the second wire. This induction is due to a part, only, of the field of stress around the first wire. The part of the field around, and due to, either conductor, and which does not cut the other wire, is called a leakage field. Due to these phenomena, there are in the medium surrounding two wires, both of which carry current, four divisions of the fields of stress, which may be considered to be separate in their effects; and the leakage electromotive force in the two wires is due to two of these fields, which are called "leakage", or "stray" fields.

The stress intensity varies hyperbolically with the distance away from, and at right angles to, the wire carrying current. (Equation 8). And the total stress included between the wire and any point outside the surface of the wire, varies according to a logarithmic function.

The total stress set up in a wire carrying a current of a given constant value, is constant, and is independent of the radius of the wire, varying directly with the current, if the current changes, (Equation 15a); also, the stress intensity within the wire varies rectilinearly, at any point, with the distance of the point from the center of the wire, (Equation 15b).

If a number of wires are placed side by side in a plane and a current sent thru them, the distribution of intensity along a line in the plane, perpendicular to the wires, may be assumed to be rectilinear, if the points considered are at the center of the space between two adjacent wires, or at the conductors, themselves, provided that the wires are fairly widely separated, and that the current flows in the same direction thru all of them. (Equation 20) If the wires are very close together, so that the distance between their centers is less than two times the diameter of one of the wires, this rectilinear distribution is very closely approximated. (Equation 20a) In the first of the two arrangements, there are lines of stress encircling each wire, independently, as may be shown both mathematically and graphically. In the second arrangement, the lines of stress encircle only the center conductor in the layer, independently as shown in Figure 24. This theory of the distribution of stress density may be applied to more than one layer of conductors; but the accuracy decreases as the number of layers is increased.

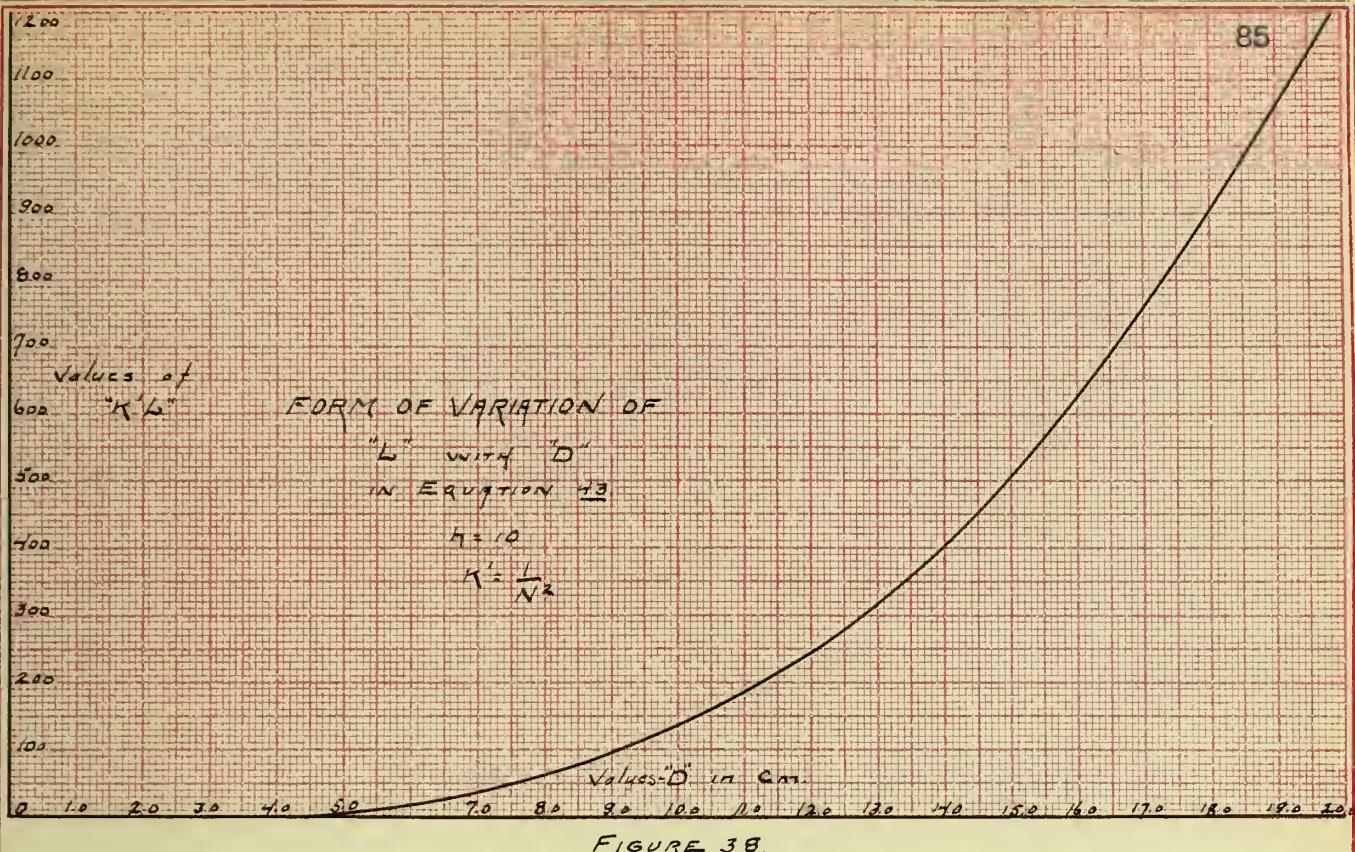


FIGURE 38.

The general theory may be applied also to a conductor wound in the form of a coil of a small number of layers; an increase in the number of layers, producing an inaccuracy, as above. By the application of this theory, it may be shown that the intensity of stress along the axis of a coil varies according to an arc tangent function. (Equation 27). As the curve of Equation (27) shows, the intensity is very nearly constant at points near the center of the coil; but, as the ends are approached, the intensity decreases due to the leakage of the lines of stress. From the outside of the coil to the center along the line of a radius, the intensity varies directly with the distance from the outer surface of the coil, according to the equation (35).

By means of the development to this point, it is shown that the total stress set up across right sections of the coil and within it, may be determined, as in equations (36) and (38). Carrying this method farther, an approximate formula for the inductance

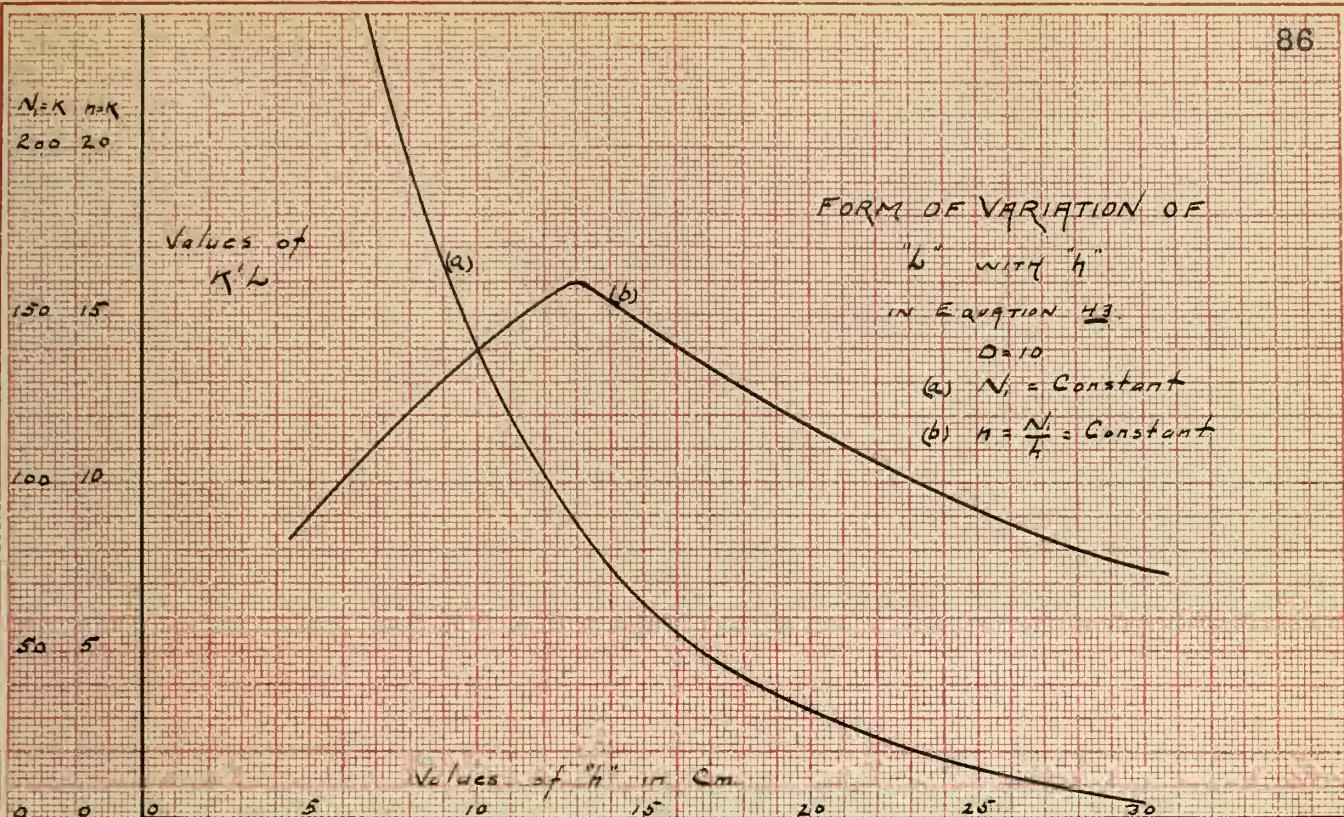


FIGURE 39.

of an air core coil may be obtained. (Equation 43; Figures 38 and 39). The curves of this equation show that the inductance of the coil decreases as the length of the coil increases, according to approximately, an exponential function, provided that the total number of turns is constant. If the number of turns per unit length is a constant, then there is a pronounced maximum in the inductance curve, at a point where the height or length is approximately 2.6 times the radius. If the radius is varied, holding the total number of turns, and the length, constant, the inductance varies according to a cubical function.

By a continuation of this method, a formula for the leakage inductance of two concentric coils may be determined. (Equation 52; Figures 40 and 41). This formula is of theoretical interest, only; since it cannot be applied to the conditions of the actual transformer. The curves of this equation show some interesting

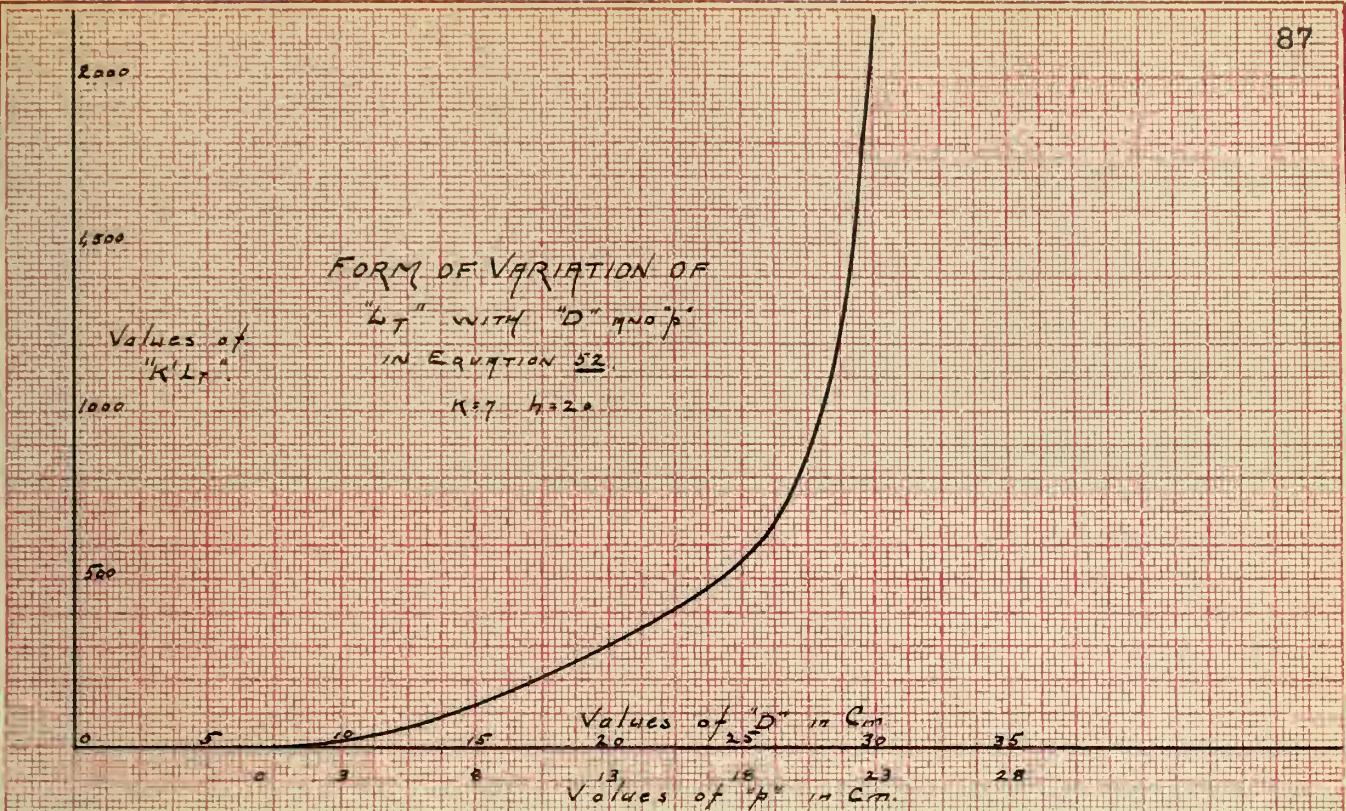


FIGURE 40.

characteristics, however. If the total number of turns is constant, the leakage inductance varies with the radius, very slowly, until a point is reached at which the radius equals approximately one half the length of the coils, after which the curve is much steeper. If the total number of turns is constant and the length is increased, the leakage inductance decreases. If the number of turns per unit length is constant, the inductance increases with the length.

It is to be noted that the equations determined in the development, to this point, are exceedingly complex from the point of view of practical calculation. Since for coils containing iron, these equations become very much more complex, different methods are employed for the calculation of, and for the development of the theory for, actual transformers.

In order to properly develop this phase of the theory, it was necessary to make a number of assumptions. These, in every

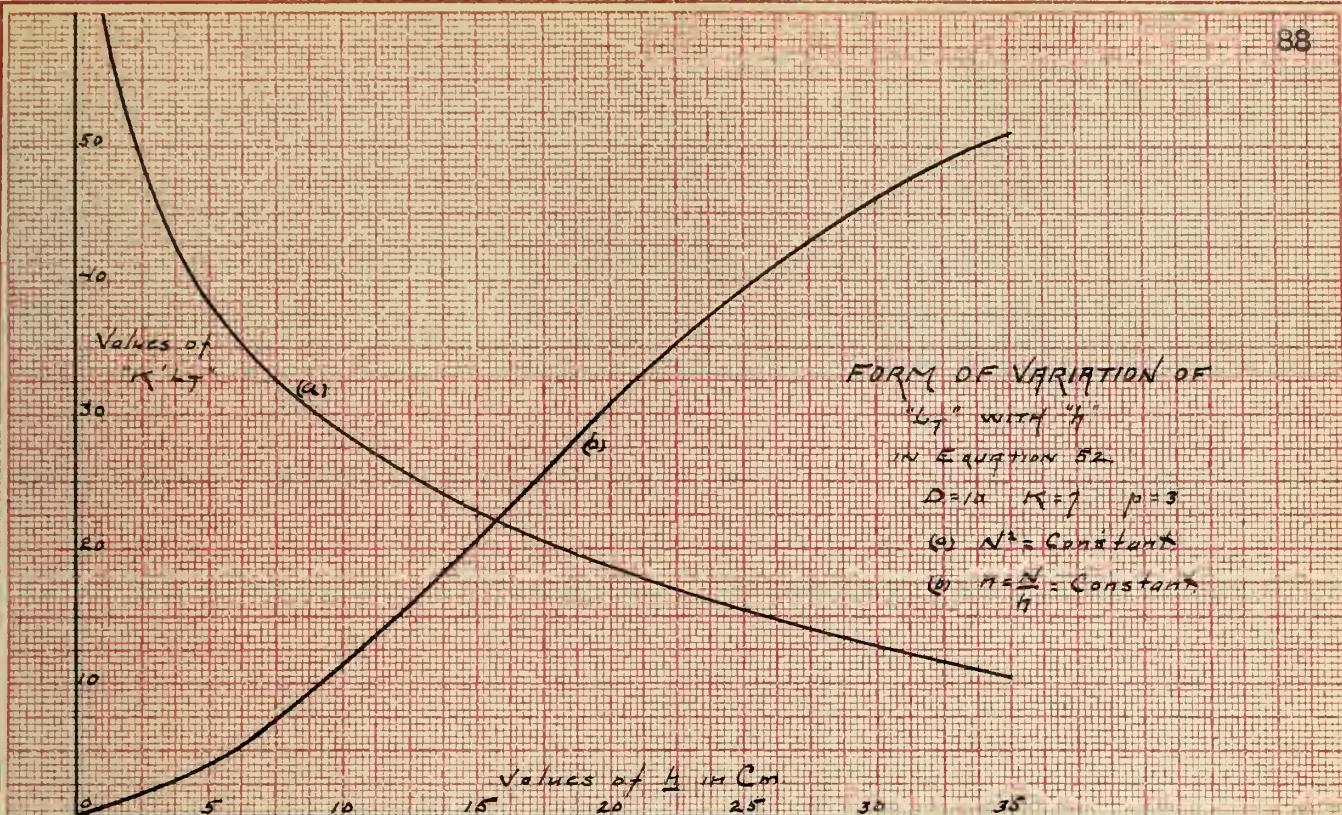


FIGURE 41.

case, have been based upon experimental observations of the characteristics, or, upon the theory developed earlier in the discussion. It is believed that the calculated values obtained from the formulae, which were developed according to this part of the theory, will check within an allowable percentage, with the actual values of leakage inductance. The great difficulty of obtaining accurate observed data sufficient to determine the true leakage inductance of a given transformer, makes the necessity for absolute accuracy in the developed formulae, very low, in practical consideration.

The results of the development are given in equations (59), (65), and (72). Equation (59) gives the result for the core type transformer with cylindrical coils. The curves of this equation (Figures 42 and 43), show that, if the diameter of the outer coil is increased, the leakage inductance increases slowly at first, and more rapidly after the radius has become equal to half the length of the

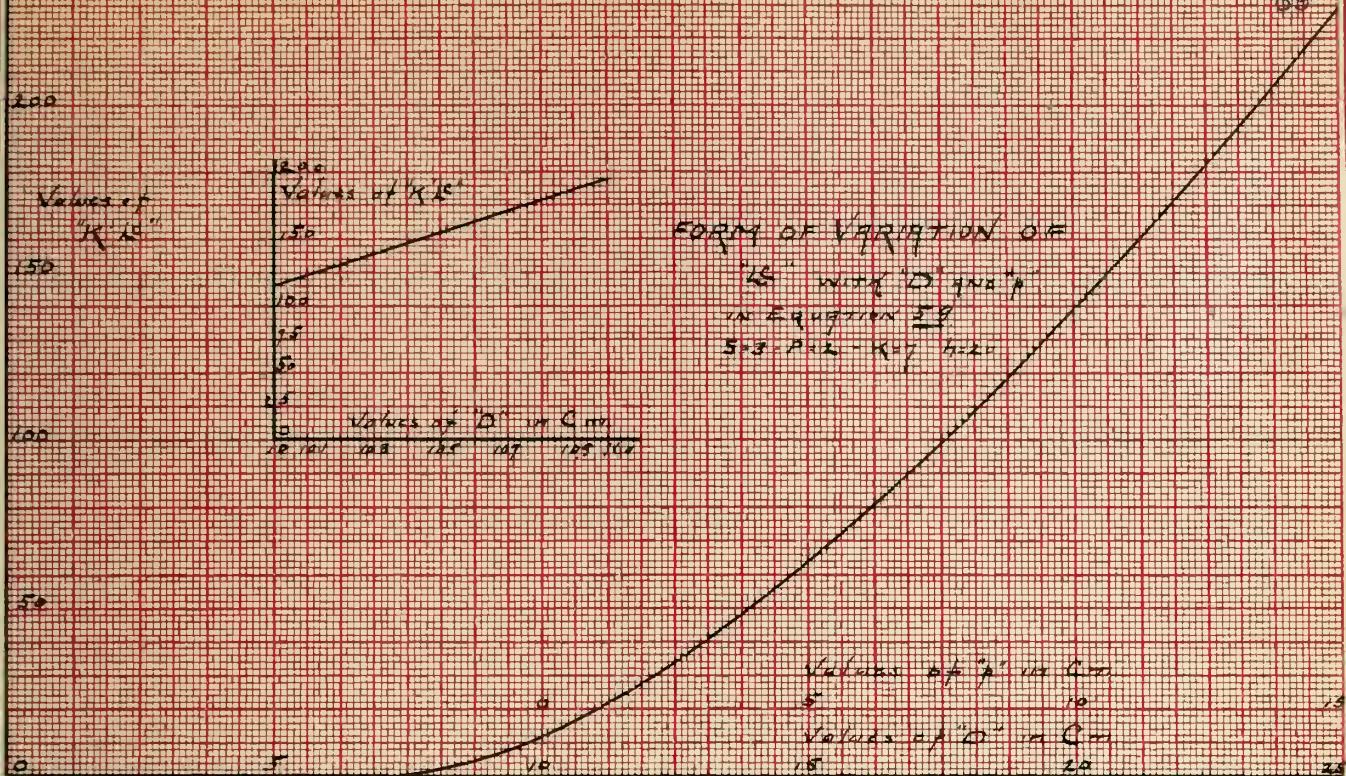


FIGURE 42.

coils. For very small values of the distance between the coils, this variation may be represented practically, by a straight line, which does not pass thru the origin, since the leakage inductance due to the flux thru the coils, is a very appreciable quantity. If the length of the coils is increased, with the total number of turns constant, the leakage inductance decreases, and if the number of turns per unit of length is held constant, the leakage inductance increases, rectilinearly, with the length of the coils.

For the shell type transformer having cylindrical coils, the results are given in equation (65). The curves for this equation (Figures 44, 45, and 46), show that, with the increase of the distance between coils, the inductance increases very nearly in direct proportion. If the length of the coils is increased, and the total number of turns is held constant, the inductance decreases with the increase in length. If the number of turns per unit length is held

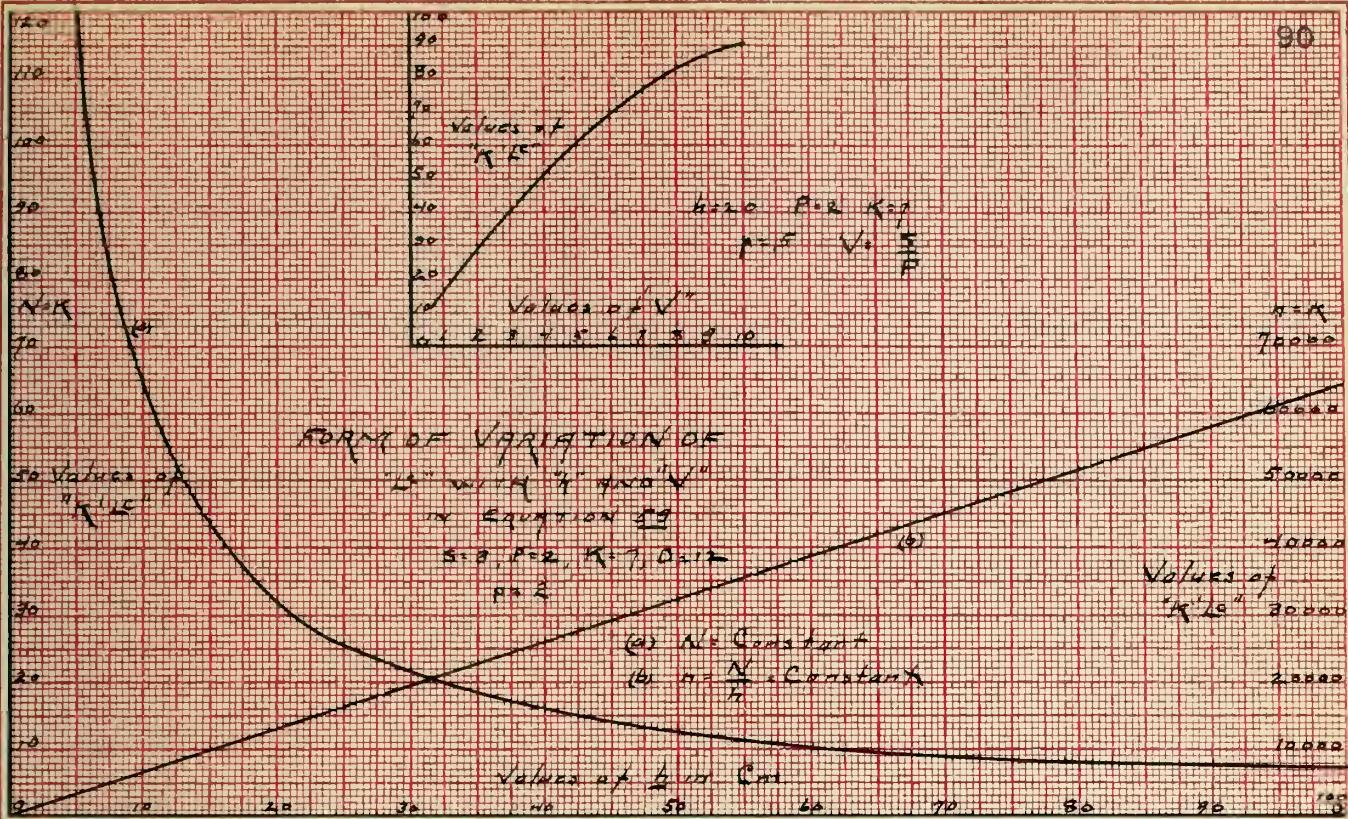


FIGURE 43.

constant, the inductance increases rectilinearly with the length. As the thickness of the coils increases, keeping a constant ratio of transformation, the inductance increases, slowly at first, and more rapidly as the thickness becomes more nearly equal to the length of the coils.

The results for the core type transformer with flat coils, are given in equation (72), (Figures 47 and 48). The most interesting point to be noted is that, as the number of coils increases, the total number of turns being constant, that is, as the subdivisioning of the winding increases, the inductance falls rapidly at low values of the number of subdivisions, but much more as this number becomes larger, showing a loss of economy in too large a number of subdivisions of the winding. As the thickness, or radial length of the coils increases, the inductance increases in very nearly direct proportion. As the distance between the coils increases,

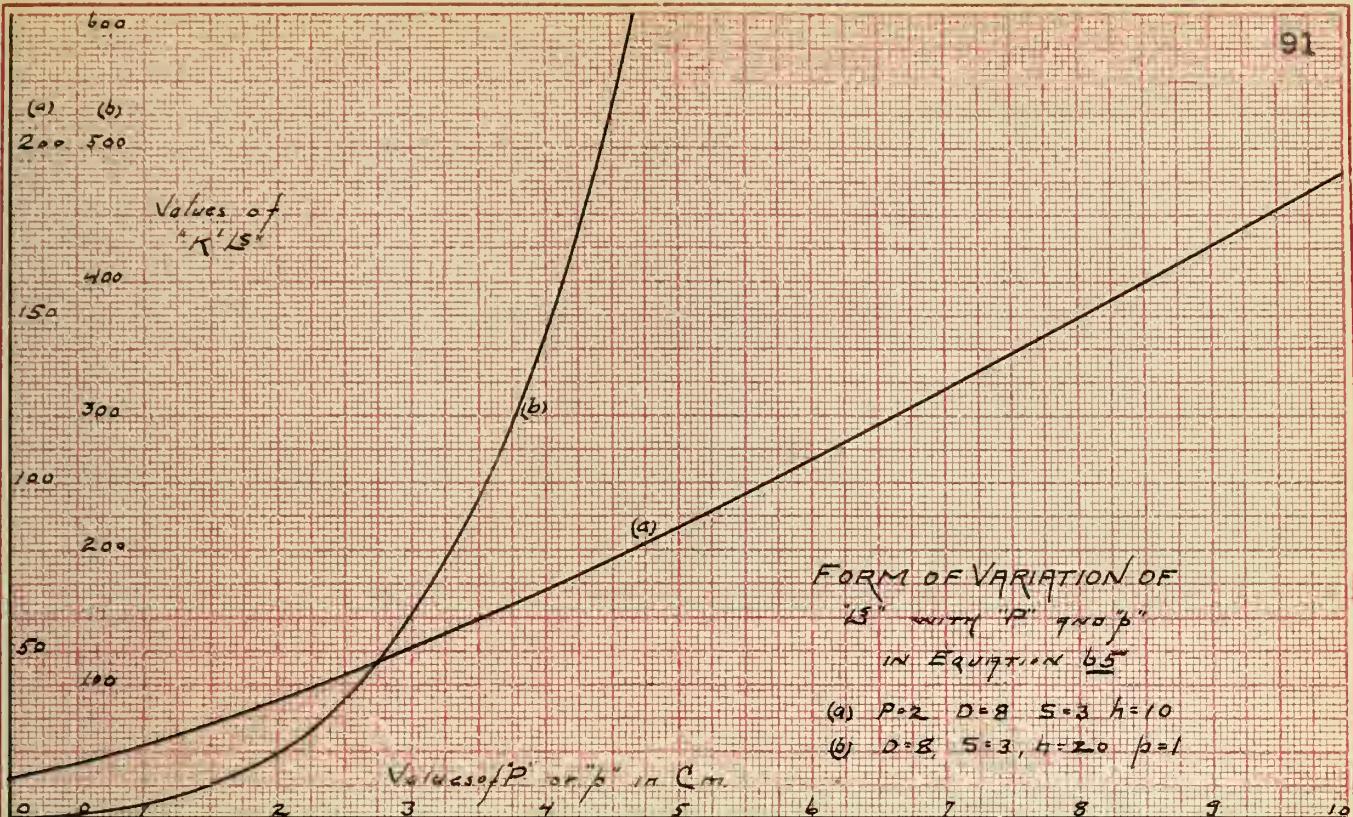


FIGURE 44

the inductance increases in direct proportion to this distance. As the true length of the coils increases, the ratio of transformation being held constant, the inductance increases slowly at first, and afterward very rapidly.

The relation of the voltage drop in the transformer, to this leakage inductance may be shown by a number of conditions. Among these, are the short circuit and open circuit secondary conditions of the transformer. It can be shown, as in equation (78), that the impressed voltage at short circuit, is the sum of the potential "drops" in the transformer due to the leakage inductance of the primary and secondary, provided that the copper losses are very small. If the copper and iron losses at short circuit, are considered, it still holds that the potential drop is the impressed electromotive force, and is that due to the leakage fields, alone. (Equations 83 and 84) Under open circuit conditions, the impressed voltage on the

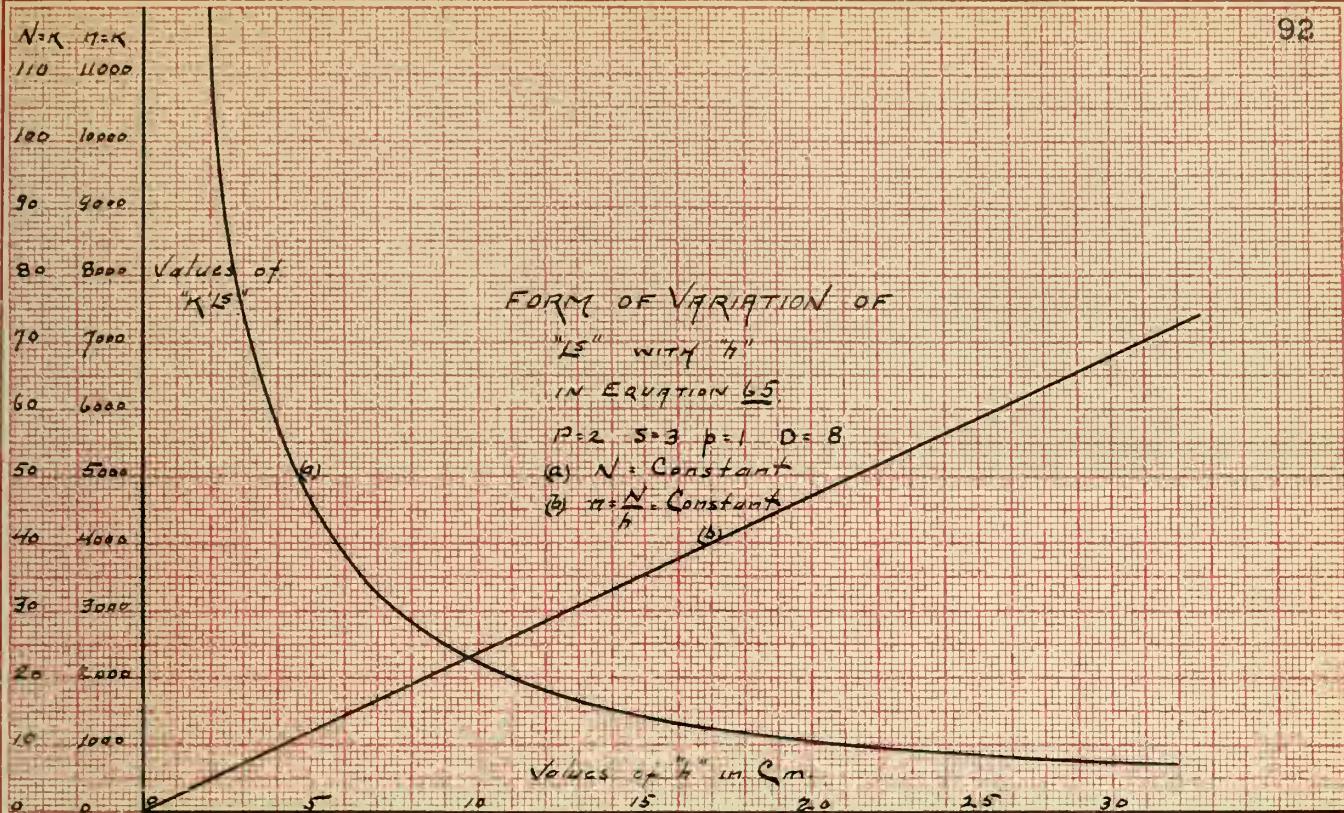


FIGURE 45.

primary, and that on the secondary, is the potential drop due to the primary stray or leakage field. These equations are not absolutely accurate, because of the difference in the leakage path at short circuit, from that at normal operation. This difference however, for all practical cases is negligible.

* * * * *

E X P E R I M E N T A L

The determination of the accuracy of a large amount of the foregoing theory depends upon experimental work which, of necessity, must be very carefully done. In order that results may be obtained, upon which reliance can be placed, and from which very nearly absolute conclusions can be drawn, it is necessary to employ instruments of very delicate construction, and which are capable of giving accurately, values of very small potentials. Now, the static voltmeter is subject to inaccuracies which make it practically useless for this purpose. Also, the instruments of the magnetic type, requiring, as they do, a flow of current for their operation, become inaccurate, and the results obtained by their use, are of no value. After a careful consideration of the results obtained by certain investigators of the leakage phenomena, who employed instruments of these types in their research, it was decided to make only qualitative tests by means of such apparatus as it might be possible to obtain, and to reserve the quantitative experiments until special apparatus might be constructed, which would be suitable for the accurate determination of points in question.

In making these tests, investigations were made, first, of the leakage lines around an iron circuit. Then the leakage between two coils on an iron circuit, was investigated. Finally, certain transformers were employed in determining some general characteristics of the leakage inductance. In describing these tests and the results obtained from them, a brief description of the instruments, and of the methods, employed will be given first; then the actual results will be discussed; and finally, the conclusions obtained

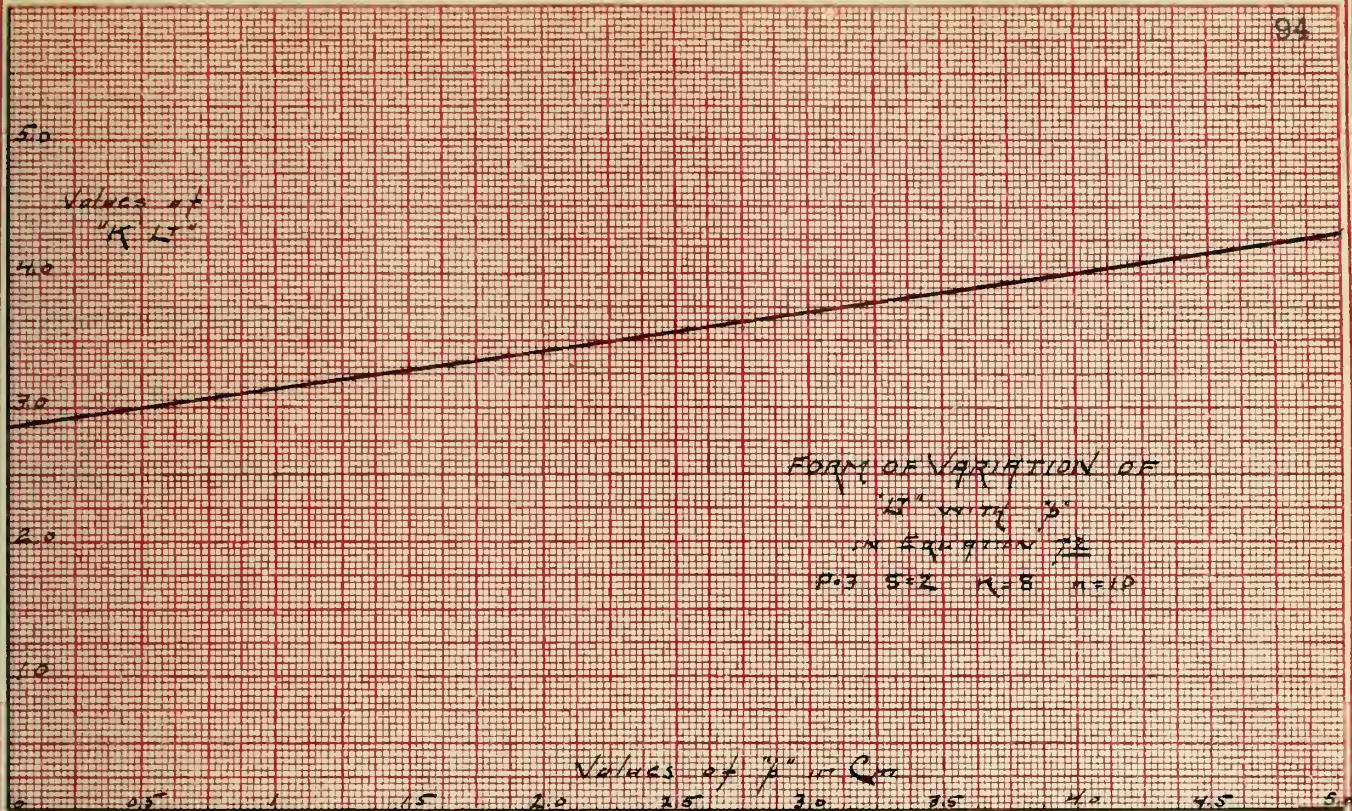


FIGURE 46.

from the experimental results will be noted.

The iron circuit used was one originally built for demonstration purposes. A general drawing is shown in Figure 50. The iron of the transformer is laminated, and the separate lamina are carefully bolted together, giving a very rigid construction to the several parts. These parts consist of the two legs A and B, the base C, and the yoke D, having the dimensions shown in the drawing. The yoke and base are fitted with small dowel pins, so placed that they correspond to small holes in the centers of the ends of the legs A and B, thus insuring the same relative positions of the parts of the transformer at all times during its use. Since the parts of the transformer are separable, an air gap is necessarily introduced at each meeting point around the iron circuit. However, careful investigation resulted in the fact that the results of the tests were in no way affected by the presence of these slight ob-

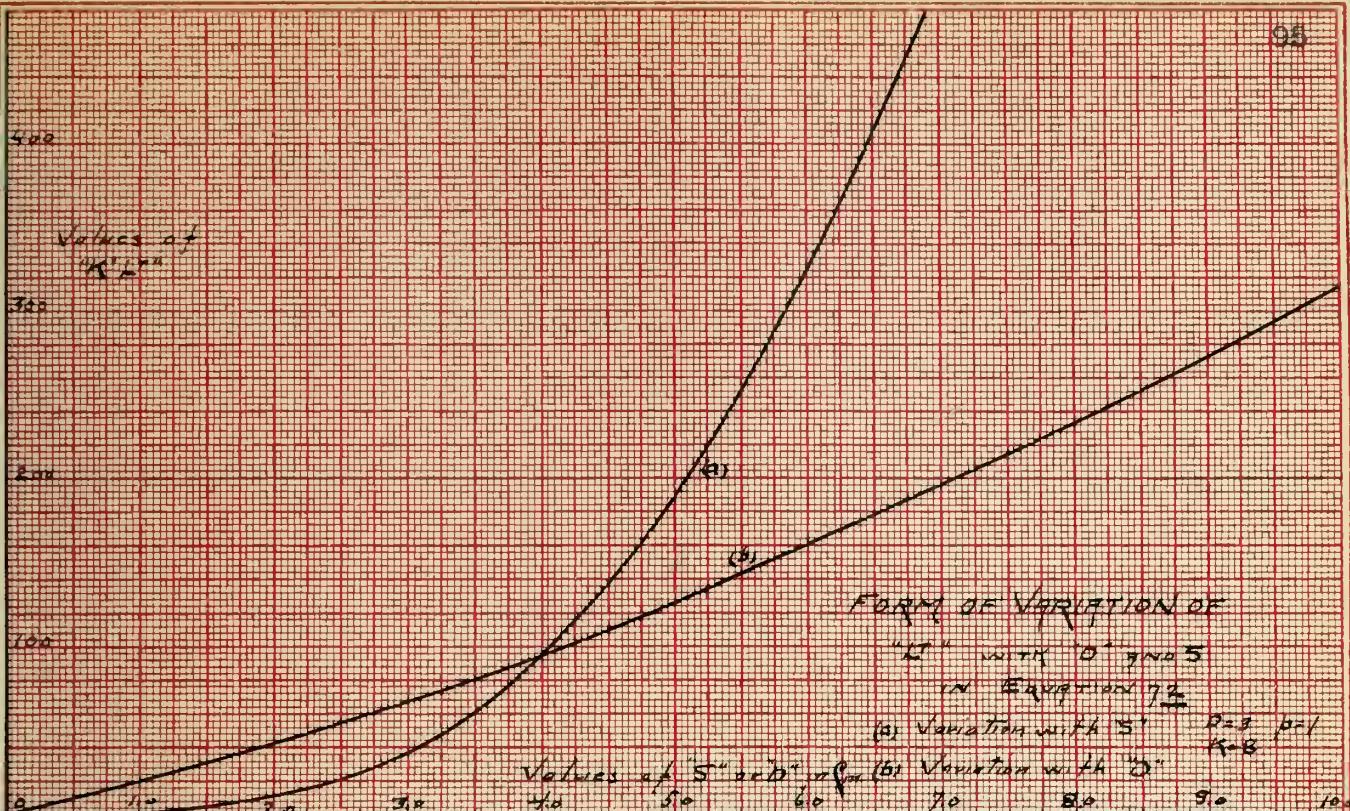


FIGURE 47.

structions in the path of the lines of force.

A number of coils were built for use with this transformer. Of these some were small bunched windings and others were larger and in the form of cylindrical coils. The small coils were wound using #18 and #14 wire. About one hundred turns were wound in each coil, since with thirty to forty amperes, this number of turns gave normal working density in the iron. The form was rectangular, and the coils were made so that they could be slipped over a leg of the transformer, leaving practically no air space between the coil and the iron. These were the ones using the large wire. Test coils, wound from the #18 to #22 wire, were so made that they could be slipped entirely across the iron circuit, from the bottom of one leg, across the yoke, and down to the bottom of the opposite leg. The cylindrical coils were wound with seventy-five turns of #14 wire, each, and also seventy-five turns of #22 wire, which was

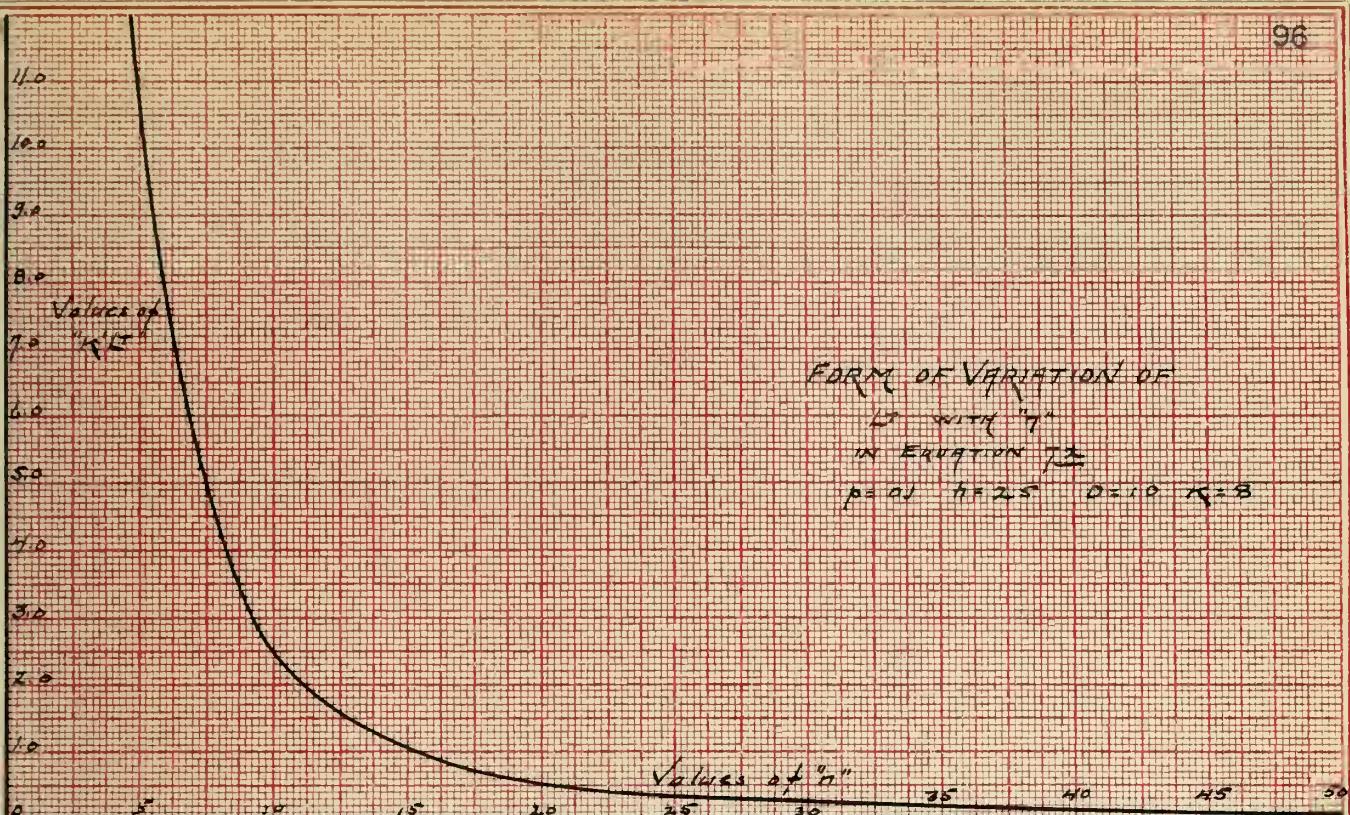


FIGURE 48.

wound side by side with the larger conductor. Each coil had but one layer of each size of wire, the larger wire running thru from one end to the other, and the smaller wire being tapped every fifteen turns to form five distinct test sections on the coil. Six coils were made in this way, having diameters as given in Table (I). An idea of the general aspect of the coils can be obtained from Figure 51.

In making tests with these coils, it was necessary to employ some instrument for fairly accurate measurement of the induced voltages in the test coils. For the first set, using the small bunched windings, an ordinary voltmeter was used for this purpose, since the induced electromotive force was large. When however, it was desired to determine the voltages induced in the cylindrical test coils, a voltmeter was not applicable and one vibrator of a General Electric Company oscillograph was employed. The resulting readings, obtained by photography, were, comparatively, very nearly

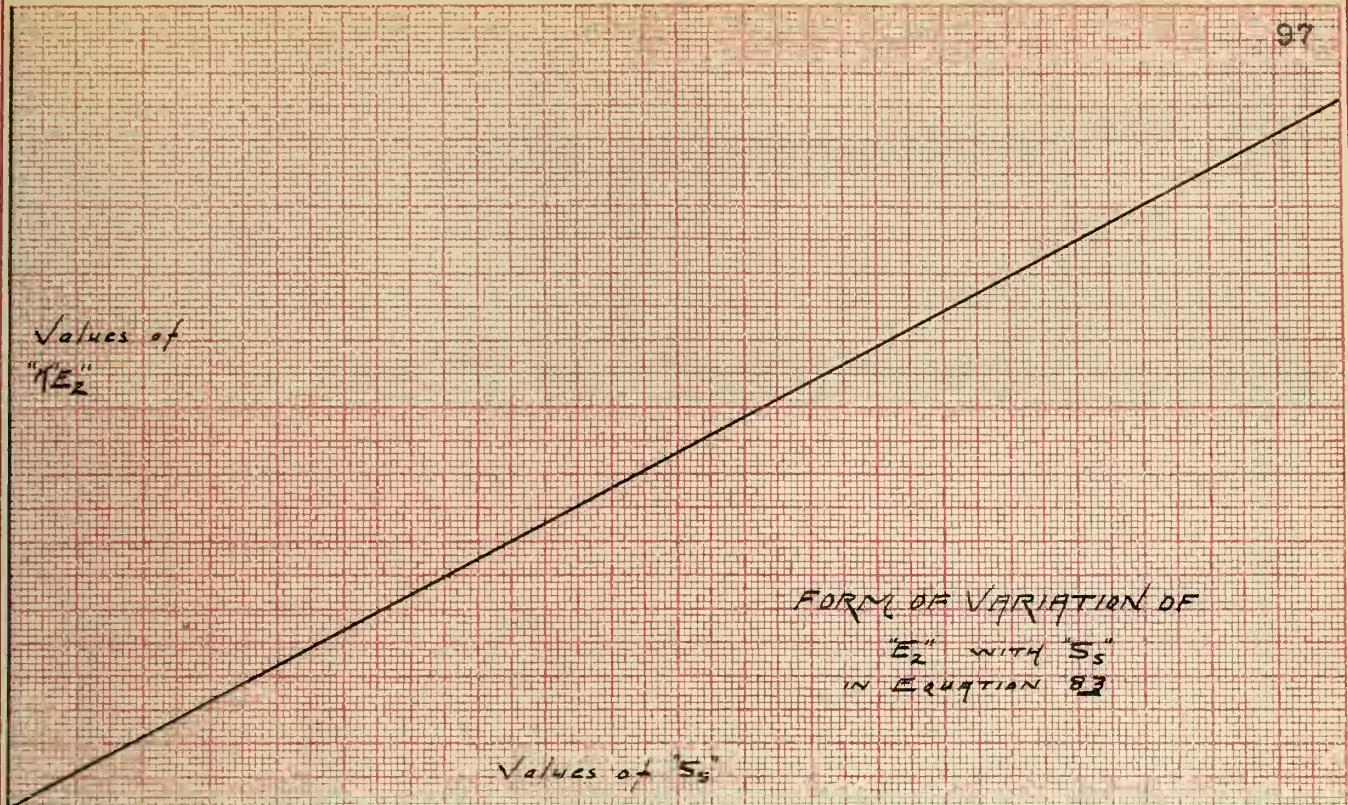


FIGURE 49

accurate.

In order to obtain an alternating electromotive force for the proper excitation of the coils, it was found best to employ a belted motor-generator set, having rather fine rheostat control in the direct current motor field. This set was driven, partly from the University power plant, and partly from a second motor-generator set, which in turn, was driven from the power plant. In order to avoid sudden fluctuations in frequency, a large amount of the work was done at night.

To measure the impressed voltage from this motor-generator set, a number of meters were used, as before mentioned. These were voltmeters of General Electric manufacture, ammeters, and a Biddle frequency meter. The voltmeters were carefully calibrated against each other, and all readings were reduced to conform to the calibration curve. The frequency meter was most necessary, since

FIGURE 50.

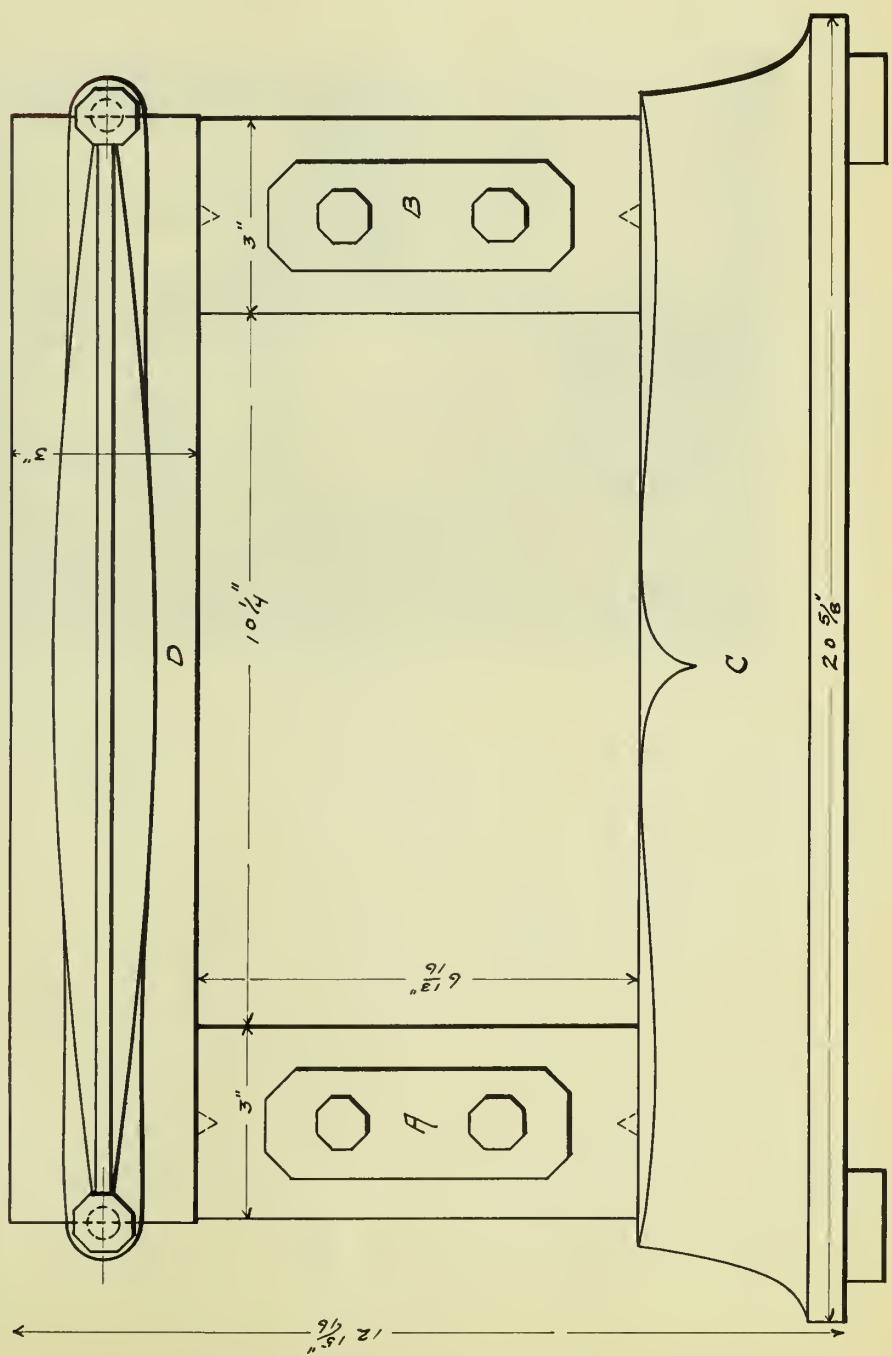
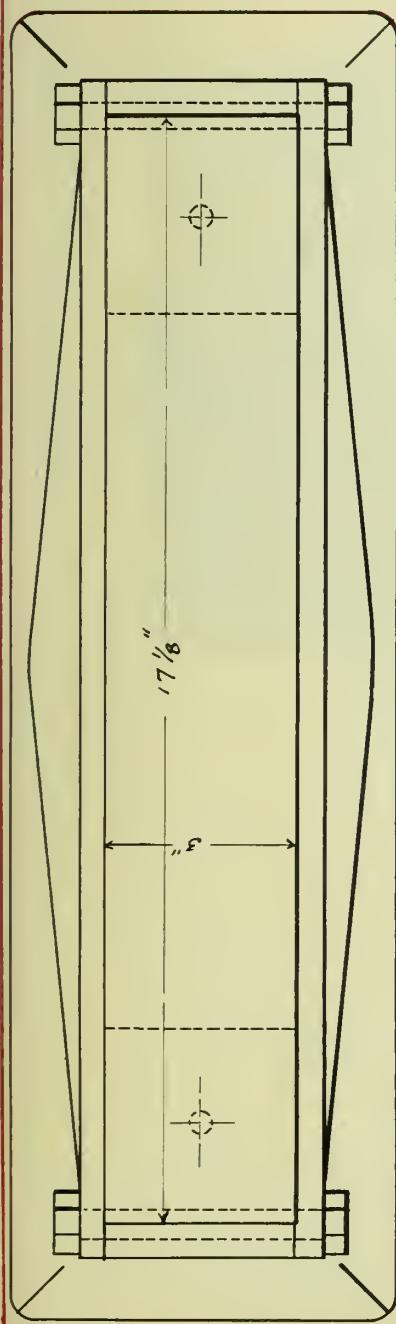
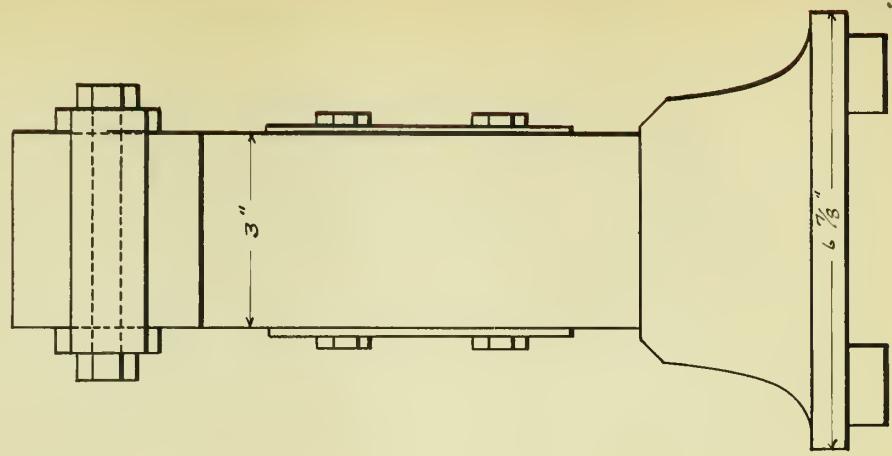




FIGURE 51.

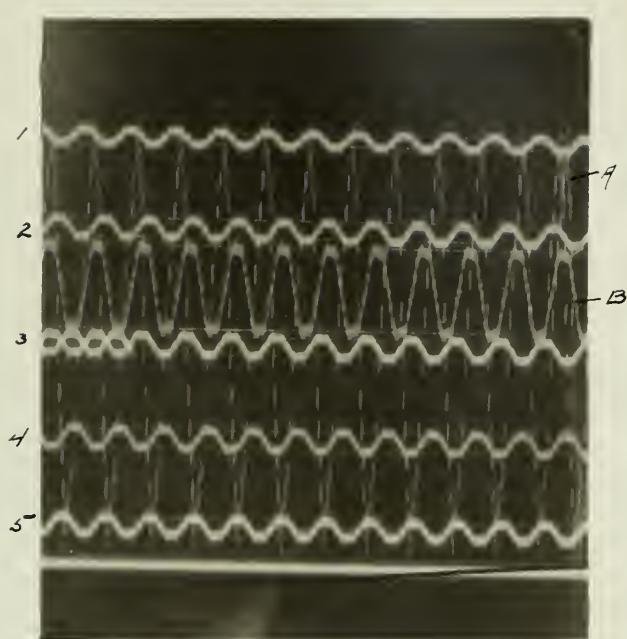


FIGURE 52

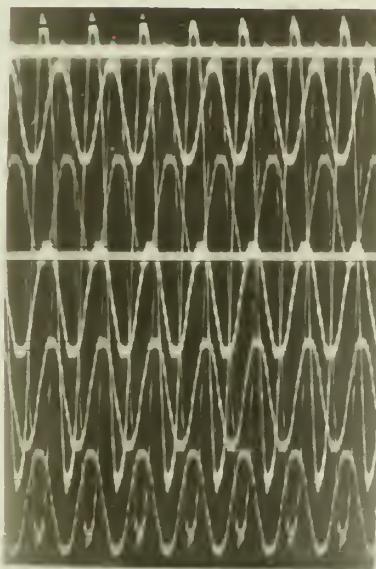


FIGURE 53.

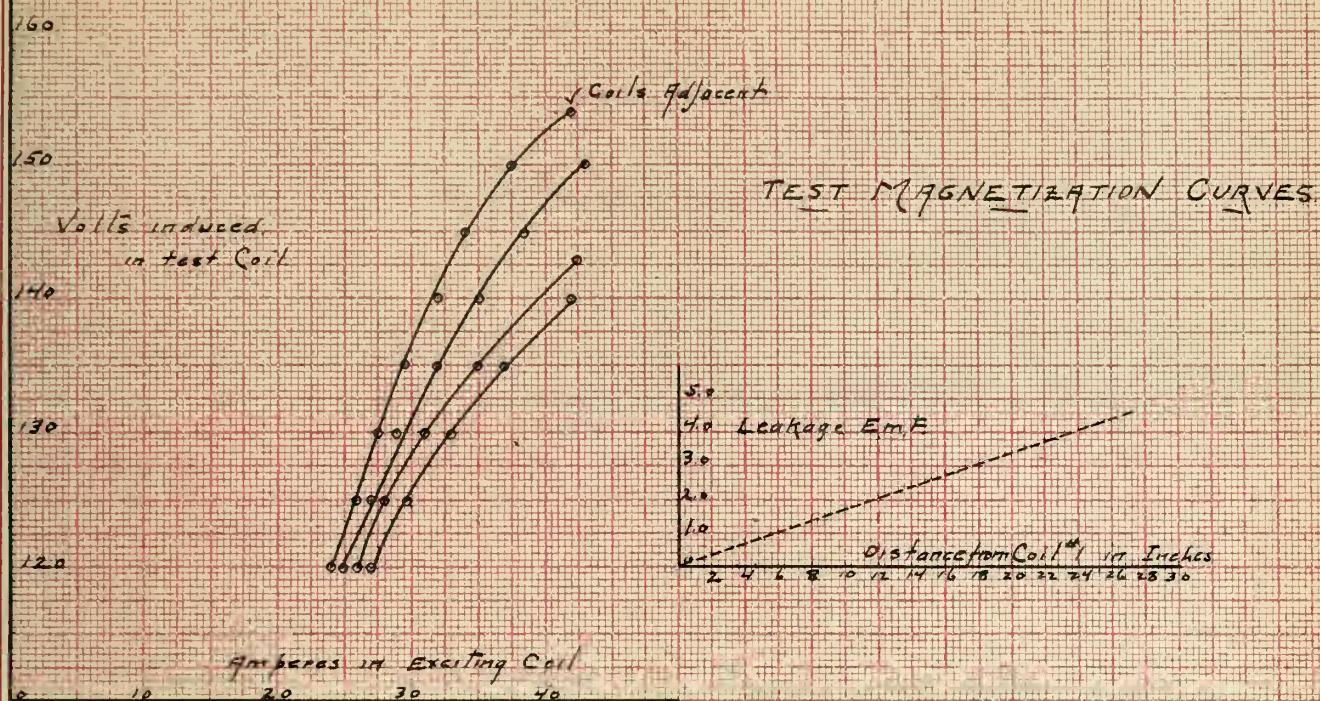


FIGURE 54

small but disastrous fluctuations of the frequency would have occurred at inopportune times, if it had not been used. It may be noted here, that certain lamp banks were used in the course of the tests, and that the voltmeters on the side of the impressed voltage, were connected across the coils alone, and not across any of the lamp banks which may have been in series therewith.

After the preliminary tests with the coils mentioned above, some general data was obtained from tests with two 1.5 k.w., core type, trans formers, of General Electric manufacture; 120/240 to 1200/2400 volts.

In the general run of the tests, the following operations were made, and the results obtained as shown----

One of the small coils of #14 wire was placed at the bottom of one of the legs of the experimental transformer, and was electrically connected, thru an ammeter and a lamp bank, with the

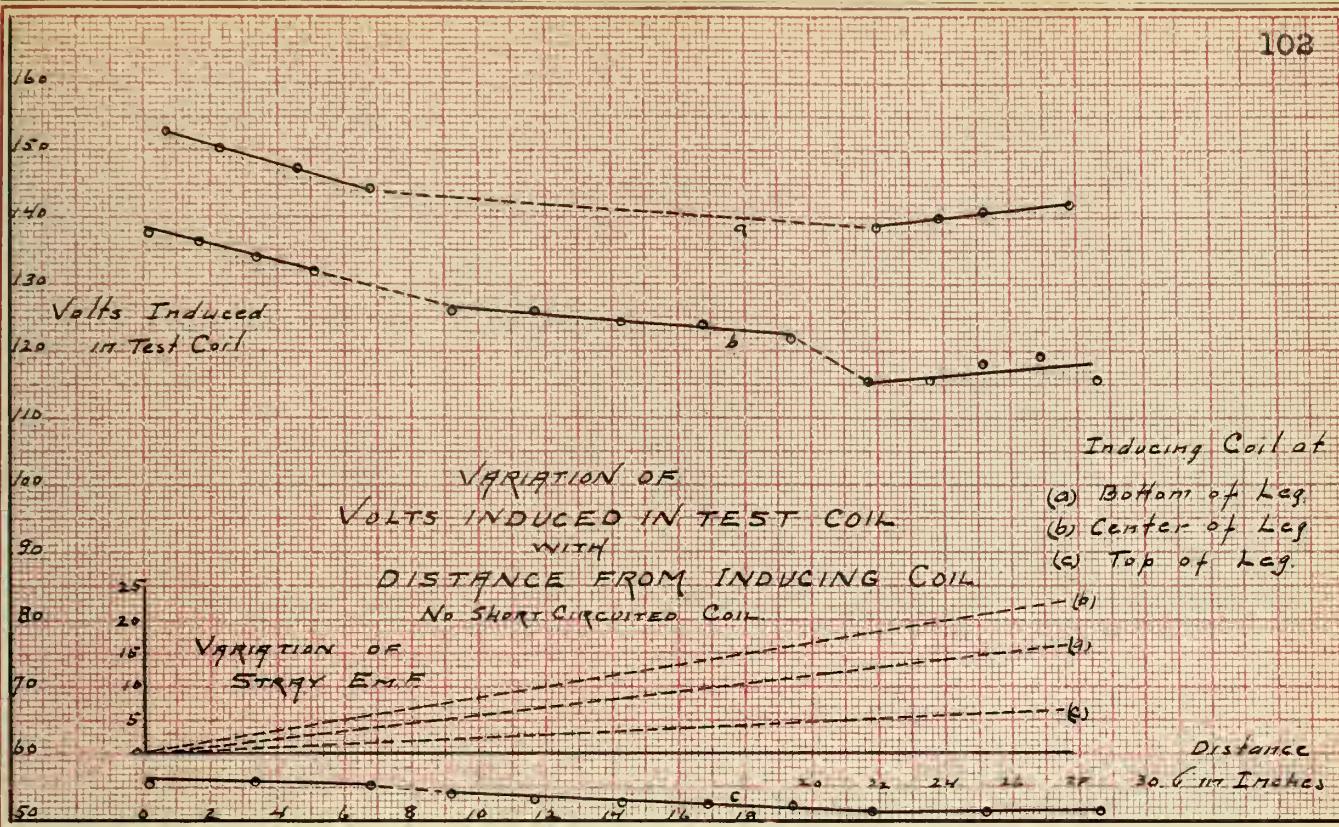


FIGURE 55

terminals of the alternating current generator mentioned above. A test coil of #18 wire also, was placed on the iron circuit and connected to a voltmeter. This latter coil was placed in various positions around the iron circuit successively, and for each position, a series of readings was taken, of the current thru the large stationary coil and the electromotive force induced in the small test coil. The result was a series of saturation curves, thru the point of normal operation of the transformer. These results are shown in Table V, and in Figure 54.

Next, the larger one of the two coils above mentioned, was placed in the center of one leg, and the small test coil was moved from a position adjacent to it, around the iron circuit. The current thru the large coil was held constant, and for each of a number of positions of the small coil, a reading was taken, of the voltage induced in the small coil. The distances were measured along

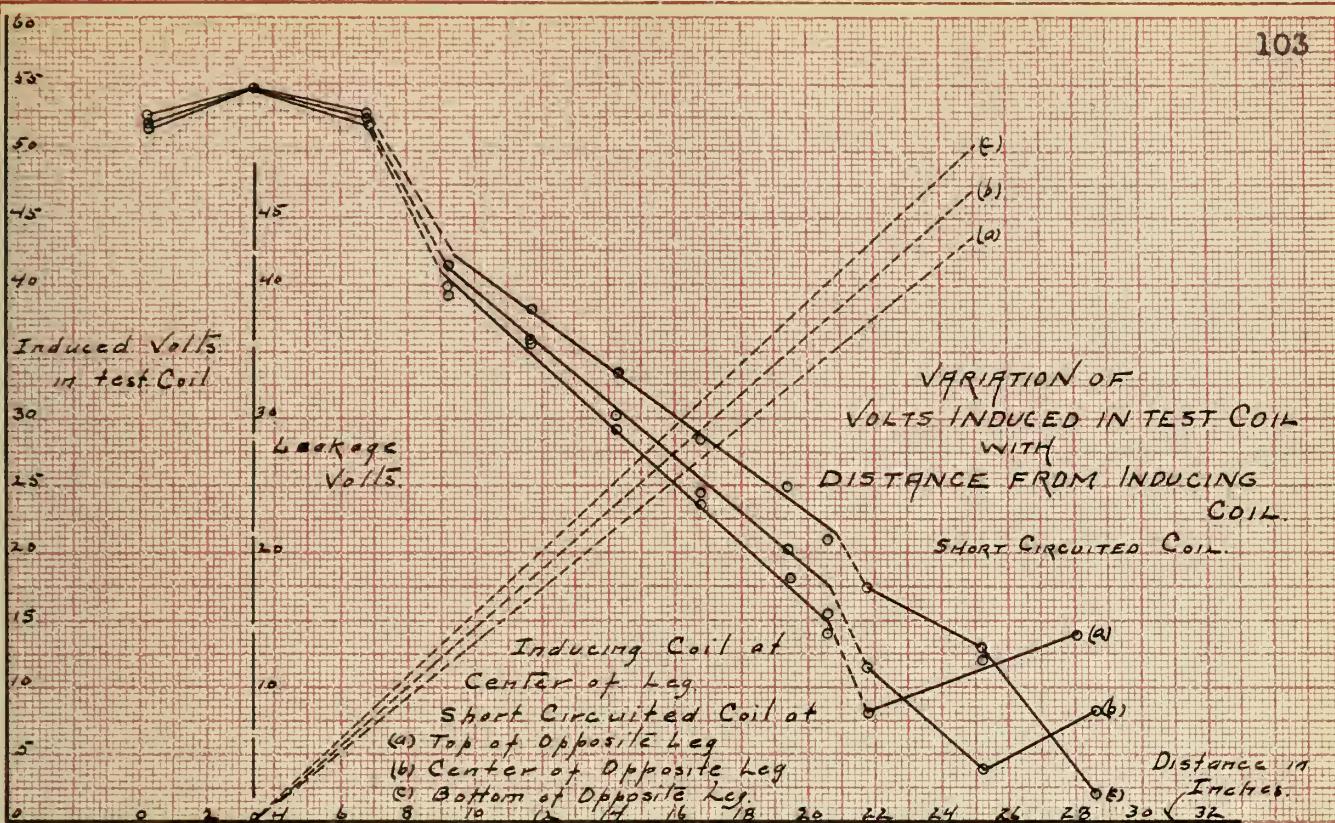


FIGURE 56

the center line of the iron circuit, from the bottom of the leg upon which the larger coil was placed. Due to the fact that the test coil used in this case, was not large enough to permit its being slipped around the corners of the iron circuit, the above operation was repeated a number of times, with a somewhat larger test coil. The results of these operations are given in Tables VII and VIII, and in Figure 55.

Another coil of the same general nature as the larger one of the two above, was added, in the next tests, as a short circuited secondary coil. The primary, or inducing, coil was placed in the center of one leg, and the secondary coil was placed at the bottom, at the center, and at the top, of the other leg, successively. For each one of these positions of the secondary coil, readings of voltage induced in the test coil at several positions of the latter around the iron circuit, were taken. The results of this series are

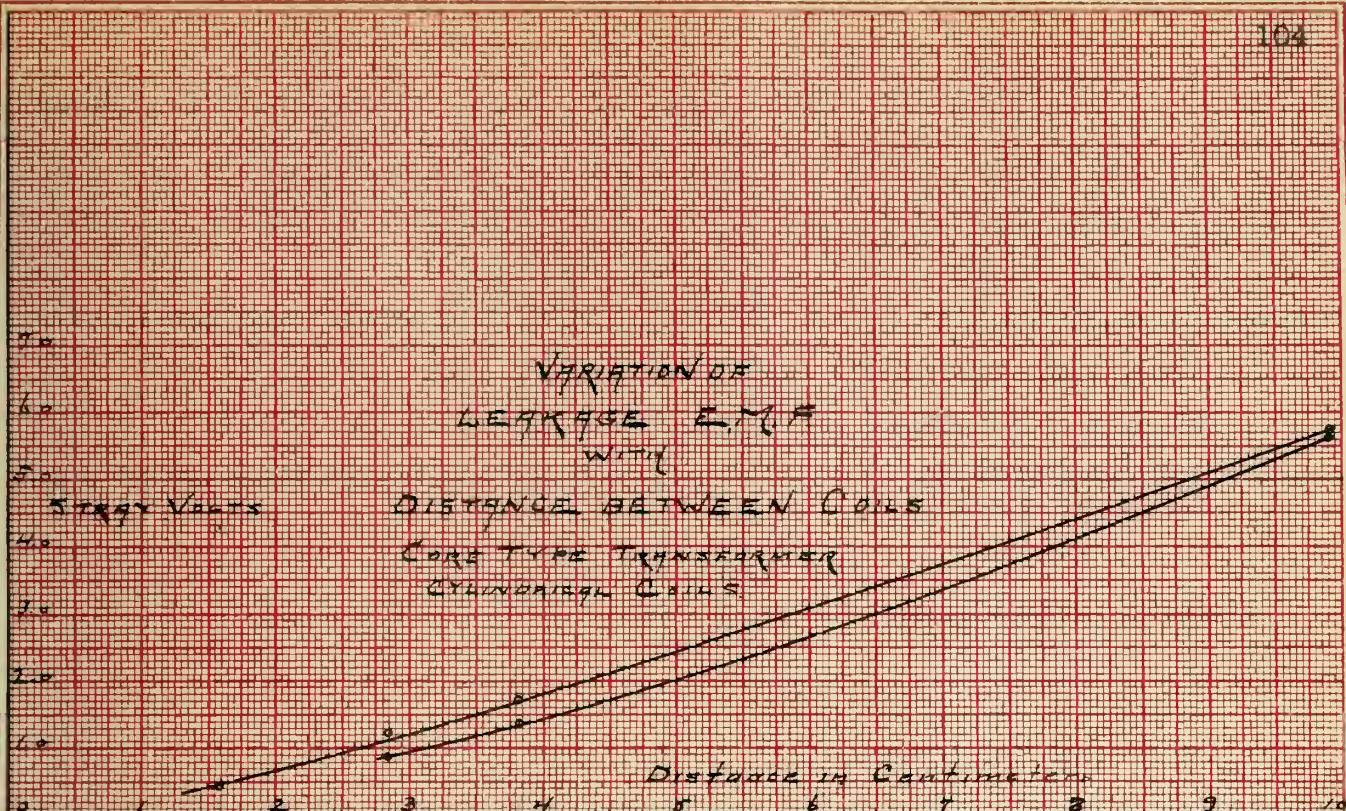


FIGURE 57

in Table IX, and in Figure 56.

For the succeeding tests, the cylindrical coils were used. Two were employed at a time, and for the first series, both were placed on one leg of the transformer, leaving the other leg free. They were electrically connected so that their magnetic effect was the resultant of the opposed effects of the coils considered separately. One part of an oscillograph film was used for the readings taken from each coil, and upon each film were recorded, the electro-motive force waves taken across both coils together, (the calibration curve), across each one of the five small test windings on the large coil, and finally, across all the small windings in series. Two films were thus taken for every combination of the cylindrical coils, taken two at a time, except that of the coils numbered 4 and 5, the diameters of which did not differ sufficiently to permit the "set up". A specimen film is shown in Figure 52. A is the cali-

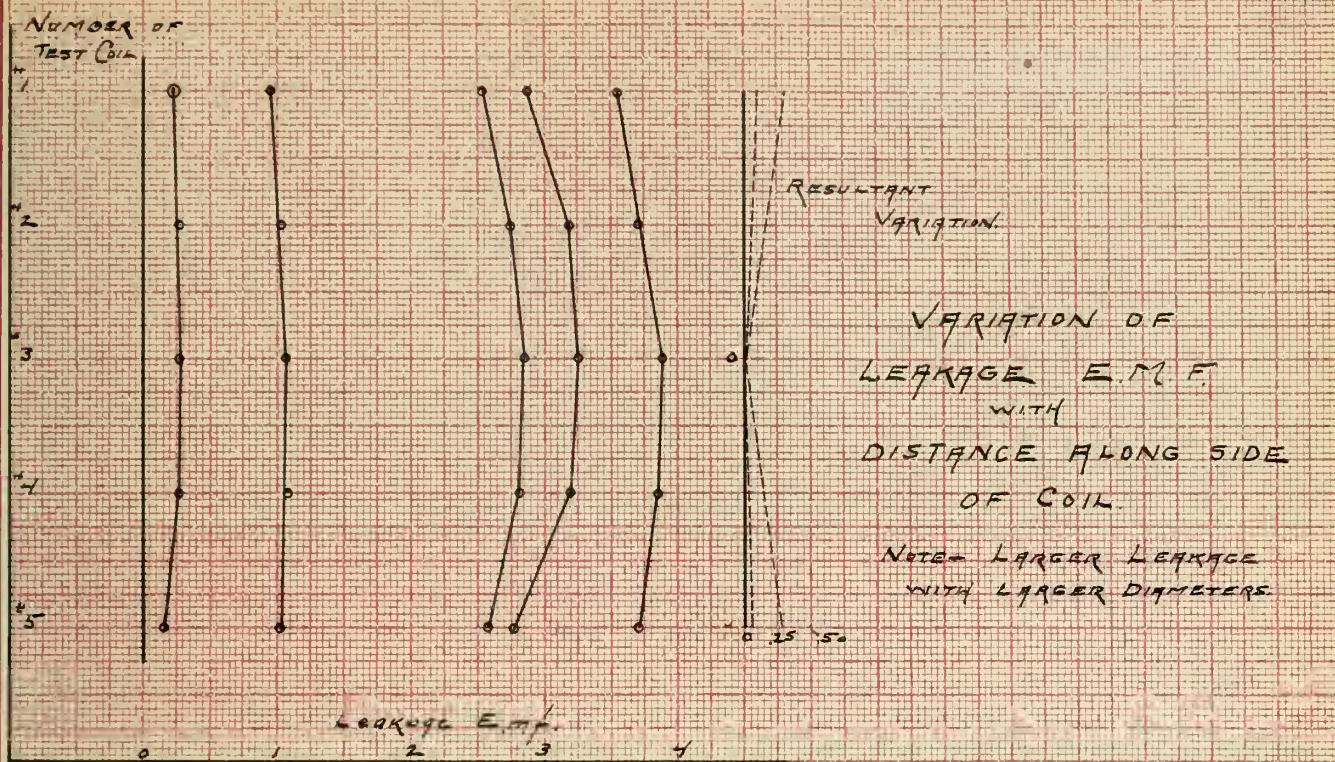


FIGURE 58

bration curve, 1, 2, 3, 4, 5, the readings from the small test windings, and B is the curve for the sum of these latter readings. From the results of this series of tests, curves were drawn showing the variation of the leakage voltage, (the sum of the B curves for any set up), with the distance between coils. Also curves were plotted showing the leakage variation along the length of the coil. These curves are shown in Figures 57 and 58; and the observed data is given in Tables III, IV, and VI.

Using the same cylindrical coils, a second series of tests was run, with one coil on each leg, and connected as above. Relatively corresponding readings were taken by means of the oscillograph, as above. A specimen film is shown in Figure 53. These results gave practically the same data as did those of the series just preceding.

After these tests, the two transformers mentioned above,

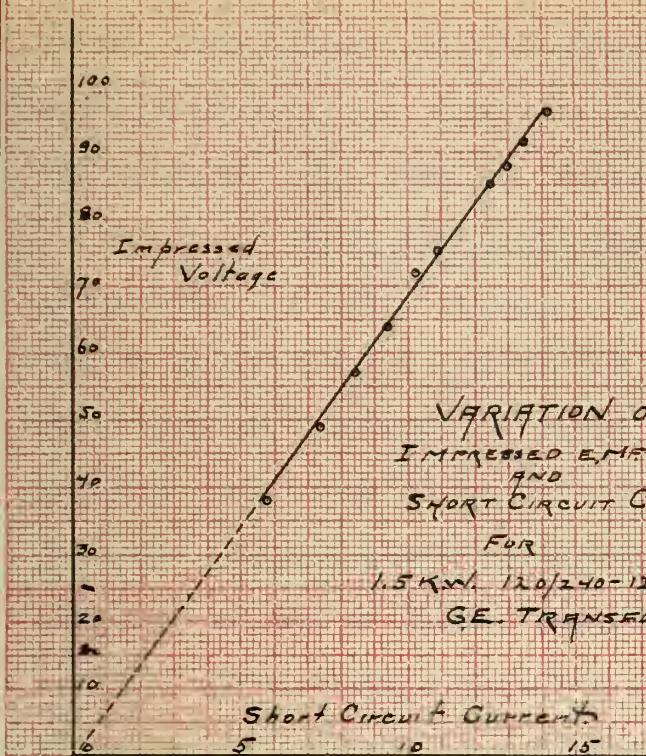


FIGURE 59

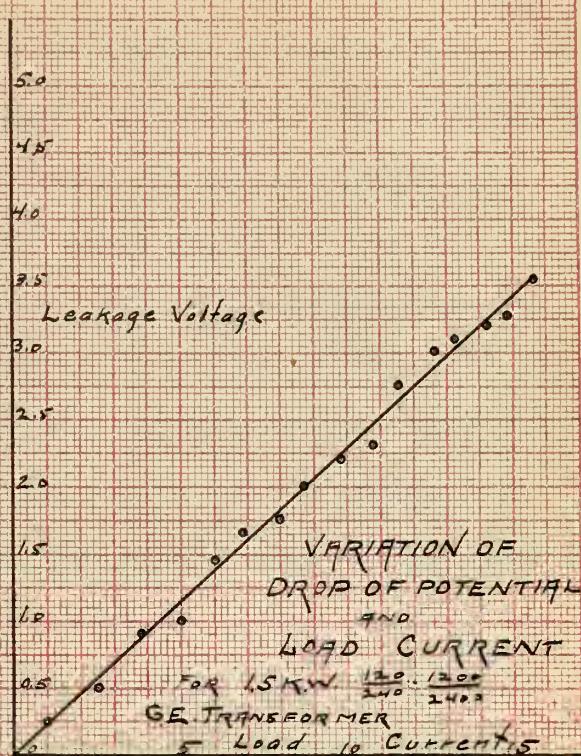


FIGURE 60

were connected with the high tension sides together. The low side of one was then connected to normal alternating electromotive force, and the low side of the other transformer was connected, thru an ammeter, to a noninductive load. Voltmeters were connected at the low tension terminals of the two transformers. The load was then varied from zero to above full load on the transformers, and for each step in the variation, readings of the load current and of the terminal voltages were taken. One half of the difference between these voltage readings for any given load value, is the leakage voltage of each transformer, provided that the resistance drop is negligibly small. A curve was plotted between the load current and this leakage voltage. The results are shown in Table XI, and in Figure 60.

Next, one of the transformers was short-circuited thru an ammeter and a low voltage was impressed upon the high tension side. Readings of this voltage were taken as the ammeter reading

varied from almost zero to above full load value. A curve was plotted showing the change in the voltage impressed as the current was changed. The results are given in Table X, and in Figure 59.

As before mentioned, the conclusions which may be drawn from the above results of the tests, are all qualitative, or are of only relative importance for quantitative comparison. Of the first series, the results offer conclusions which consider only the paths of the lines of magnetic stress thruout the arrangement, or "set up" for any given condition. From the second series, certain conclusions may be drawn, regarding the behavior of the leakage inductance when certain definite changes are made in the arrangement of the coils. While in the last tests of the actual transformer, only general conclusions regarding the variations of the leakage reactance may be drawn.

A brief consideration of the curves resulting from the first series of tests, (Figure 55), shows that as the test coil was moved around the iron circuit, and away from the inducing coil, the leakage increased. And the auxiliary curve shows that this relation is practically rectilinear. From this curve and those following, of this series, the lines of stress seem to follow the general paths shown in Figure 61. While, to a certain extent, this is the usual conception of the stress field, one or two interesting points are to be noted. The leakage of lines from points around the iron circuit seems to be comparatively, very small, under this arrangement, (that is, without a short circuited secondary coil) The leakage across the corners is not noticeably greater than that at other places around the circuit, even including the air gap at the points where the legs join the yoke and base. This is due in part, to the

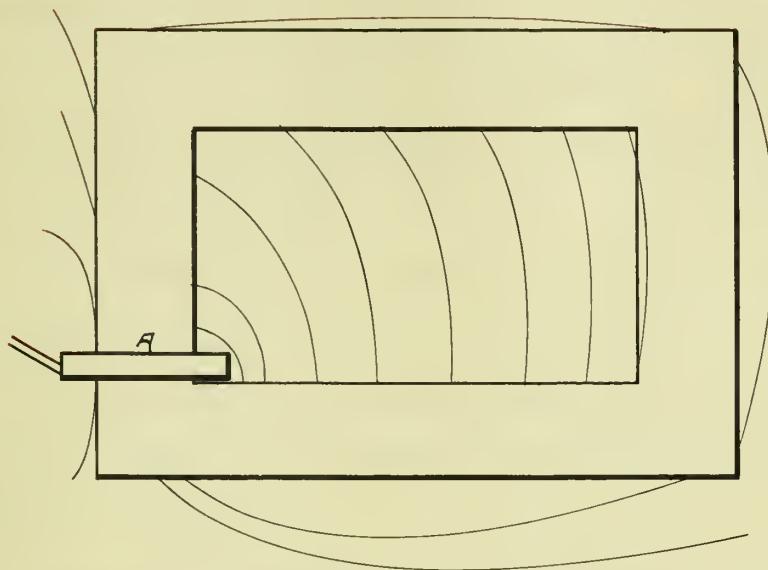


FIGURE 61

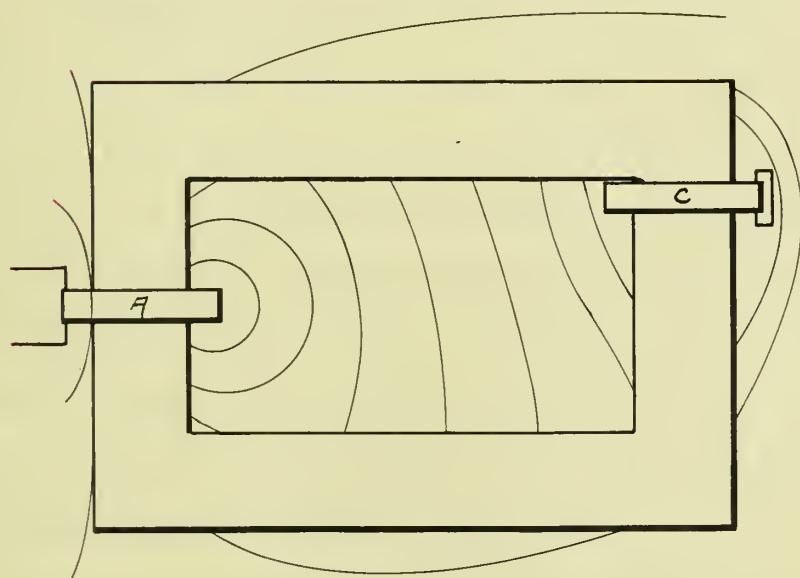


FIGURE 62

comparatively low flux density at which the test was made.

The curves resulting from the test using a small short circuited secondary, (Figure 56), show a larger leakage from different points around the iron circuit, due to the fact that the magnetic effects of the two coils were in opposition. The general form of the field around the iron circuit, is that shown in Figures 62 and 63. The coils c are short circuited. By the theory of stresses in such a coil, there are no resultant lines of stress thru it, provided that the voltage drop due to resistance is negligible. In these cases this drop was practically negligible and the lines pass as shown in the figures. As before, the leakage along the yoke of the iron circuit is uniform.

In the results from the cylindrical coils, it will be noted that the leakage from the sides of a coil having an iron core, varies directly with the distance from the center of the coil. The number, and the position, of the points in the curves would permit drawing a line thru them, which is not exactly straight under all circumstances; but the deviation from a rectilinear curve is very small and in practical consideration, can be neglected. The variation curve of the leakage voltage with the distance between coils is, in proximity to the iron, not a straight line; but the curve has an increasing slope after leaving the immediate vicinity of the core. For the coils farther from the iron, however, this curve becomes practically rectilinear. (Figure 57)

The tests with the two General Electric transformers are interesting in one point at least. From the curve in Figure 59, it may be concluded that the leakage inductance is practically constant for all loads. Also, the other curve obtained shows that

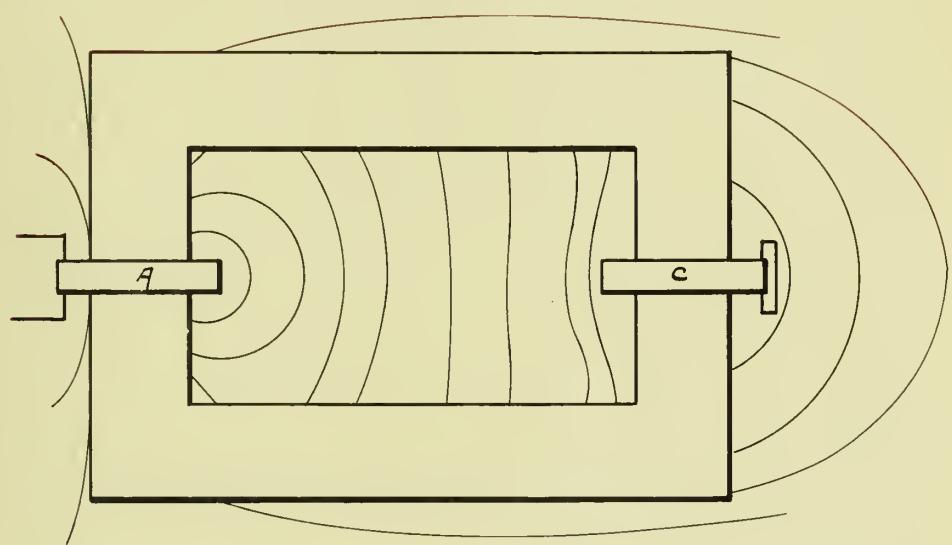


FIGURE 63

for low values of impressed electromotive force, as at short circuit test conditions, the leakage inductance variation is rectilinear, although the operation is at the very lowest part of the iron curve around the lower knee as shown in Figure 37.

* * * *

G E N E R A L C O N C L U S I O N S

The general conclusions which may be drawn from the theoretical and experimental development of the phenomena of leakage inductance, given heretofore, may be divided into two classes: Those obtained from the development which was preliminary to that of the actual inductance formulae, and those obtained from the development of these formulae and of the relations of the leakage inductance to the other general characteristics of the transformer. Of these two classes, the second offers conclusions which apply most directly to the question of the inductance of the actual transformer; but a number of interesting general conclusions may be found in the first class, also. In giving these general conclusions, specific references to specific results will not be made, and only the most important of the conclusions given before will be repeated, since they may be found in full in their proper place at the close of the theoretical and experimental discussions.

In the first of the classes mentioned above, the following important deductions are to be noted:

The magnetic condition of a medium surrounding a conductor carrying a current, is due to stresses set up in the medium by the magnetic action of this current. The distribution of these stresses is such that they may be represented by tubes of unit stress which may be considered to exist thruout the part of the medium affected. These tubes of unit stress may be represented by lines of unit stress, or of "force" as they are usually designated.

By means of this stress theory of the distribution of

the magnetic effect of the current flowing, the phenomena of magnetic stresses about a "live" conductor, of the distribution of stresses about a number of conductors placed parallel to each other, and finally, of the distribution of stresses in the neighborhood of conductors wound in the form of coils, may be investigated mathematically. And since this is true, it follows that a mathematical development of the leakage stresses thru coils, and finally, of the leakage inductance of coils, may be obtained.

In order to develop the theory of stresses to the point of its application to the determination of the inductance formulae, it is necessary to make certain assumptions. Both the previous theoretical, and the experimental, developments show that, for all practical cases, these assumptions are justified. However, for a rigid mathematical determination of the inductance formulae, these assumptions are not justified. For example, in the case of large reactance coils, or later, that of large transformers, the distribution of the current within the large conductors is, very probably, not such as to give equal current density in all parts of the right section of one of the large conductors, and the variation of the stress intensity within the conductor, with the distance from the center of the conductor, is not rectilinear. Hence, the assumption that such rectilinear variation holds in all cases, introduces an error in the very accurate determination of the inductance by the formulae developed, when very large coils are considered. For all practical purposes with coils of the most common size, the assumption holds.

The one other very important assumption was made in the development preliminary to the inductance formulae of the actual transformer. That is, that the variation of the leakage stresses

thru the sides of a coil having an iron core, with the distance from the center, and toward the ends, of the coil, is rectilinear. The experimental results show that this assumption, also, is justified for all practical cases. Consequently, the formulae of leakage inductance of coils having either iron or air cores, can be considered to be accurate, practically, so far as their development is concerned.

In the consideration of the conclusions of the second class, applying to the transformer itself, and to the relations of the leakage inductance to the other characteristics of the transformer, it is to be noted, in general ---

That reduction in the distance between the primary and secondary coils, in the case of the cylindrical coil type of transformer, reduces the leakage inductance, and consequently, improves the regulation of the machine. This result is obtained by increasing the length of the coils of transformers having a given number of turns; in which case, however, the question of efficiency enters to a very appreciable extent.

That an increase in the number of subdivisions of the windings in the case of the transformer with flat coils, decreases the leakage inductance of the machine; but also, that when this number of subdivisions reaches a moderately high value, the proposition of increasing it still further, becomes uneconomical.

That the short circuit test of a transformer offers a valuable method of checking the calculated values of the leakage inductance of any given machine. If such a test is not possible, an approximately accurate determination may be made by means of the open circuit test. If either of these tests is impossible, due to actual use of the transformer, or some such condition, the determination

may be made, by employing two voltmeters, finding, by means of them, the voltage drop thru the machine under consideration. In all of these tests, account must be taken of the resistance drop thru the transformer if very accurate results are desired.

One very important conclusion which may be drawn from the paper in general, is that the development of any formula suitable for the exact determination of the leakage inductance of a transformer, before or after the machine is built is an exceptionally difficult problem. And that, while approximate formulae may be developed for such determination, the results obtained from their application are, by no means, absolutely accurate. It must be noted here, that the equations for leakage inductance, given in the body of this paper are not exact; but, that they are believed to be approximations which are within a permissible percent of error, for practical conditions.

In general, it may be concluded that, while a knowledge of the exact value of the reactance of transformers is absolutely essential for their proper operation with the highest economies possible, it is nevertheless true that such knowledge must depend, either upon the results obtained from formulae, or upon those from actual tests. If it is possible to properly operate the machines from a knowledge results only, then the formulae obtained as are the ones given above, are sufficiently accurate for practical conditions. However, if an exact knowledge of the inductance of such machines is necessary for their production, sale and operation, the practical question of the investigation of the leakage phenomena or of the leakage inductance of transformers is by no means solved; the opportunities for deeper investigation, are as much in evidence as they have been heretofore; and, "the time is not yet come".

A P P E N D I X

In the general determination, by means of certain experiments, of the characteristics of the lines of magnetic stress, a large amount of data was taken, most of which offered some valuable conclusion, or at least was instrumental in producing the general conclusions from the experimental part of the paper. A number of the tests were the same; and in some of them, a great amount of the data taken offered results along certain lines which repeated themselves. Since this is the case, the final results are practically assured. Also, since this is the case, it is necessary to give here, only sample data, or specimens of certain parts of the general data taken; where a compilation of the whole amount would be extremely bulky, and would not be of value. It is the purpose of this appendix to offer such specimens of the data, in the form of the following Tables, as are necessary for the proper understanding of the discussion of the experimental conclusions. Since reference is made in this discussion to all these tables, it is not necessary here to repeat the significance of the data given. The Tables follow:

Table I.

No. of coil	1	2	3	4	5	6
Diam. in cm.	11.74	13.64	14.92	17.46	19.38	31.40
Radius in cm.	5.87	6.82	7.46	8.73	9.69	15.70

Table II.

Distance between cylindrical coils in cm.

No. of coil	1	2	3	4	5	6
1	0	0.95	1.59	2.86	3.82	9.83
2		0	0.64	1.91	2.87	8.88
3			0	1.27	2.23	8.24
4				0	0.96	6.97
5					0	6.01
6						0

Table III.

Total observed leakage voltage

No. of coil	1	2	3	4	5	6
1	0	3.30	3.54	3.19	3.23	6.05
2		0	0.48	1.24	1.76	5.75
3			0	0.85	1.38	5.65
4				0	0	4.77
5					0	4.44
6						0

Table IV.

		Total observed leakage voltage.				
Combination.	No. of coil	Number of test coil.				
		1	2	3	4	5
1 and 6	1	1.94	2.07	1.99	1.99	1.88
	6	3.57	3.70	3.87	3.87	3.66
2 and 4	2	2.63	2.92	2.96	2.93	2.72
	4	2.76	2.98	3.05	3.00	2.82
2 and 6	2	2.52	2.74	2.83	2.78	2.56
	6	2.86	3.16	3.22	3.20	2.72
5 and 6	5	2.73	2.90	2.90	2.91	2.73
	6	2.82	3.06	3.12	3.12	2.86

Table V.

Data for test saturation curves.

Induced voltage.	Distance of position from primary, In.			
	0	4.6	22.0	25.2
120	24	25	26	27
125	26	27	28	32
130	27.5	29	31	33
135	29.5	32	35	37
140	32	35	--	42
143	--	--	42.5	--
145	34	38.5	--	--
150	37.5	43	--	--
154	42	--	--	--

Table VI.

Data for leakage thru sides of cylindrical coils.

No. of Curve	Number of test coil				
	1	2	3	4	5
1	0.24	0.25	0.26	0.25	0.15
2	0.95	1.02	1.05	1.07	1.02
3	2.55	2.75	2.85	2.80	2.60
4	2.83	3.20	3.25	3.20	2.75
5	3.55	3.70	3.90	3.85	3.70

Table VII.

Distribution data for small test coils.

Distance along iron.	Induced Voltage.	Distance along iron.	Induced Voltage.
0.7	153.0	0.3	55.2
2.3	150.7	3.4	55.6
4.6	147.5	6.8	55.2
6.8	144.7	9.2	53.9
23.0	138.4	11.7	53.6
23.9	140.0	14.3	53.0
25.2	141.0	16.8	52.6
27.8	142.1	19.4	52.2
		20.6	51.8
		21.8	51.5
		25.2	51.4
		28.6	51.5

Table VIII.

Distribution data for small test coils.

Distance along iron.	Induced Voltage.	Distance along iron.	Induced Voltage.
0.2	138.0	14.3	124.1
1.7	136.2	16.8	123.9
3.4	133.9	19.4	122.0
6.8	130.0	21.8	118.0
9.2	126.0	25.2	118.6
11.7	126.0	28.6	118.2

Table IX.

Distribution data for small test coils.

Distance along iron.	Position of primary on leg.		
	Bottom	Middle	Top
0.2	52.0	52.2	52.6
3.4	54.9	54.9	54.6
6.8	52.9	52.2	51.8
9.2	41.4	40.1	39.2
11.7	38.4	36.0	35.8
14.3	33.4	30.4	29.4
16.8	28.4	24.4	23.6
19.4	24.9	20.2	17.9
20.6	21.2	15.6	14.3
21.8	17.5	11.4	8.0
25.3	13.0	4.0	12.0
28.6	4.0	10.6	16.5

Table X.

Data from short circuit test.

Impressed Voltage.	Current (Secondary)	Impressed Voltage.	Current (Secondary)
97.5	14.2	75.0	10.9
96.1	14.2	64.0	9.4
94.0	13.9	57.0	8.5
91.5	13.6	49.0	7.4
88.0	13.0	38.0	5.8
85.0	12.4		

Table XI.

Data from loaded transformer test.

Primary Voltage	Secondary Voltage	Secondary Current.
110.0	110.1	0.0
109.5	109.0	1.0
109.0	108.0	2.5
109.0	107.0	5.0
108.5	105.2	6.8
110.5	107.0	8.0
108.0	103.6	9.8
107.0	102.5	11.4
107.8	101.6	13.2
107.4	100.9	14.7
106.7	99.5	15.6





UNIVERSITY OF ILLINOIS-URBANA



3 0112 082197044